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ON THE TRANSMISSION OF LIGHT THROUGH A CLOUD OF RANDOMLY DISTRIBUTED PARTICLES

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1. Introduction

The study of the transmission of light through a cloud of particles is of practical importance, since it is connected with natural phenomena such as the opacity of clouds and smokes, and the visibility in fog and rain. So far, such a study has been restricted only to those cases where the particles are very small. Lord Rayleigh (1899), for instance, has used his theory of the scattering of light by small particles to determine the attenuation of the transmitted light. There is, however, no satisfactory simple theory, either for transparent or opaque particles of bigger size, distributed in a transparent medium. Mallock (1919) has attempted to give a theory of the transmission of light through a cloud of water droplets, but his theory is defective in many respects. He has assumed the droplets to be opaque, which is not true. Also, he has treated the problem on a geometrical basis, omitting the effect of diffraction, which is not justified even with opaque particles. Thirdly, his application of the theory of probability to the geometrical method is also not correct.*

In this paper, a general theory is developed of the propagation of light through a cloud of spherical particles, randomly distributed in a transparent medium. Both the cases when the particles are opaque, as well as when they are transparent are considered from the point of view of the wave-theory of optics. It is shown that in general, there is always an attenuation of the transmitted beam, which is of the exponential form. The numerical value of the attenuation coefficient has also been calculated in a number of cases. In particular, Mallock's formulæ are corrected during the course of the development of the theory.

2. Geometrical Treatment of the Problem

In this section, we shall develop a theory of the effect on the transmitted light of a number of opaque spherical particles distributed at random

* Richardson (1919) has also given a similar theory, but he too has taken the droplets to be opaque, and has not taken diffraction into account.

in a transparent medium, purely on the basis of geometric ideas. The case in study may be illustrated by the transmission of light through heavy smoke, where the particles composing the smoke can be considered to be opaque.

Suppose that a beam of cross-sectional area A traverses a column of the smoke of length l, and that N is the number of particles per unit volume of the medium. Assume further that the particles are all of the same size, and have a radius a. On a geometric basis, the effect of each particle would be to cut out a portion of the transmitted beam of area πa^2, and thus reduce the energy content by an amount corresponding to this area. In this way, each particle reduces the energy content of the beam by a certain fraction, and it would thus appear that if the number of particles n_0 be such that $n_0 \pi a^2 = \mathrm{A}$, then the beam would be completely cut out. However, it is not so, on account of the fact that some of the particles would screen those behind them, and thus increase the chance of a portion of the direct radiation coming through. It is therefore a question of probabilities to determine what fraction of the energy is transmitted by the particles in the medium. In our case, the number of particles, n, is evidently $= \mathrm{NA}l$.

The problem at hand is identical with one in which n disks, each of radius a, are thrown at random on an area A, and it is required to find the probable area covered by the n disks. Let P_m be the probable fraction of the area covered after m disks are thrown, and let p be the fraction of the area covered by a single disk when alone, *i.e.*, $p = \pi a^2/\mathrm{A}$. Now, on adding one more disk, it may fall on the area already covered, or on the area uncovered. The probability of its falling on the empty area is evidently $(1 - \mathrm{P}_m)$. Hence, the probable fraction of the area covered after $(m+1)$ disks are thrown is

$$\mathrm{P}_{m+1} = \mathrm{P}_m + p\,(1 - \mathrm{P}_m) = \mathrm{P}_m\,(1 - p) + p \tag{1}$$

Now, $\mathrm{P}_1 = p$, so that

$$\mathrm{P}_2 = p\,(1-p) + p,$$

$$\mathrm{P}_3 = p\,(1-p)^2 + p\,(1-p) + p, \text{ etc., giving}$$

$$\mathrm{P}_n = p\,[(1-p)^{n-1} + (1-p)^{n-2} + \cdots + 1]$$

$$= p\,[1 - (1-p)^n]/[1 - (1-p)] = [1 - (1-p)^n]. \tag{2}$$

Thus, the probable fraction of the area not covered after n disks are thrown is

$$\mathrm{Q}_n = 1 - \mathrm{P}_n = (1-p)^n. \tag{3}$$

Coming back to the problem of the transmission of the beam of light, the probable fraction of its area A, not cut out after it has encountered

n particles is given by (3). Hence, the intensity of the transmitted beam is

$$I_l = I_i\,(1 - \pi a^2/A)^{AN'l}, \tag{4}$$

where I_i = intensity of the incident beam. Since $\pi a^2/A$ is a small quantity, we get

$$I_l/I_i = e^{-\pi a^2 N' l}. \tag{5}$$

Thus, it is seen that the intensity of the transmitted beam is never zero, but that it diminishes exponentially with increase of the thickness of the medium. Also, from the relation $Q_n = (1-p)^n$, it is seen that the effects of the particles are *multiplicative*, each one diminishing the intensity in the ratio $(1-p):1$. This result is important, since we have started with the assumption that the effect of each particle is *subtractive*, *i.e.*, each subtracts a portion of the energy from the incident beam, corresponding to its area. However, on account of the random distribution of the particles, the effect becomes multiplicative.

Incidentally, it may be noted that if $n = 1/p$, *i.e.*, if the total area of all the particles is equal to the area of the aperture, then $Q_n = (1 - 1/n)^n = e^{-1}$. Hence, the probable fraction of the energy coming through is $1/e$. Mallock has wrongly assumed the value 1/2 for this quantity without any proof.

The actual value of the attenuation coefficient derived in this section is not correct, for, as is shown in Section 3, diffraction effects produce a doubling of its value. But, before proceeding to consider these, it will be interesting to discuss what the attenuation would be, according to geometric ideas, if the particles were not all of the same size. In this case, we must expect a distribution of size among the particles. As a general case, suppose that the number of particles per c.c., having a radius between a and $a + da$ is

$$dN = f(a)\,da, \tag{6}$$

where $f(a)$ is the function giving the law of distribution.

Now, since all the particles are not of the same size, the application of the theory of probability has to be modified. Let $p_1, p_2, p_3, \cdots$ be the fraction of the area covered by the first, second, third, etc., particles, when alone. These can be equal or different. Then, we have the general relation similar to (1), *viz.*,

$$P_{m+1} = P_m + p_{m+1}\,(1 - P_m) = P_m\,(1 - p_{m+1}) + p_{m+1} \tag{7}$$

Hence,

$P_1 = p_1$, $\qquad Q_1 = (1 - p_1)$

$P_2 = p_1\,(1 - p_2) + p_2$, $\qquad Q_2 = (1 - p_1)\,(1 - p_2)$

$$P_3 = p_1(1-p_2)(1-p_3) + p_2(1-p_3) + p_3, \quad Q_3 = (1-p_1)(1-p_2)(1-p_3),$$

etc.

$$P_n = p_1(1-p_2)\cdots(1-p_n) + \cdots + p_n, \quad Q_n = (1-p_1)(1-p_2)\cdots(1-p_n) \quad (8)$$

Thus, every one of the quantities Q_n is symmetrical in $p_1, p_2, \cdots$ so that the order of the quantities $p_1, p_2, \cdots p_n$ is immaterial. Hence, we may regard each particle as producing an effect independent of the rest, and reducing the energy of the transmitted beam in the ratio $(1-p_n):1$. Hence, expression (4) can be generalised, and written as

$$I_l/I_i = (1-p_1)(1-p_2)\cdots(1-p_n) = e^{-\sum_1^n p_n}.$$

If we denote by a_n the radius of the nth particle, $p_n = \pi a_n^2/A$. Also, on account of the distribution (6), the summation $\sum_1^n p_n$ can be written as

$$\sum_1^n \pi a_n^2/A = \pi l \int a^2 dN. \quad (9)$$

Hence,
$$I_l/I_i = \exp[-\pi l \int a^2 dN]$$
$$= \exp[-\pi l \overline{a^2} N] \quad (10)$$

where $\overline{a^2}$ is the mean square of the radius.

As a particular case, we may take the Maxwellian law of distribution given by $f(a) = a^2 e^{-c^2 a^2}$, or

$$dN = a^2 e^{-c^2 a^2}\, da. \quad (11)$$

In this case, $\overline{a^2} = \frac{3\pi}{8}\bar{a}^2$, where $\bar{a}$ is the mean radius, so that

$$I_l/I_i = \exp(-3\pi^2 \bar{a}^2 N l/8). \quad (12)$$

This expression can also be put in terms of the most probable value of the radius, a_m, as

$$I_l/I_i = \exp[-3\pi\, a_m^2\, N l/2]. \quad (13)$$

So far, we have been dealing with spherical particles. This assumption is, however, unnecessary, for the shape of the obstacle is unimportant. The really important quantity is the area of the beam that it cuts out. If we represent the area of a particle by α, then, $p = \alpha/A$, and (5) can be written in the form

$$I_l/I_i = \exp[-\alpha l N]. \quad (14)$$

In the case, where the size of the particles is not the same,

$$I_l/I_i = \exp[-l \int \alpha dN]. \quad (15)$$

3. *Transmission according to the Wave Theory*

It is well known that a small opaque circular obstacle placed in the path of a beam of light produces a diffraction pattern, whereby part of the energy is diffracted in directions away from the forward one. On account of this, it is evident that the reduction of energy in the forward direction will be greater than that due to the mere cutting off of a portion of the wave-front by the obstacle. In the following discussion, it will be shown that the reduction is exactly double of what would be expected on the geometrical theory.

In this connection, it is important to consider what we mean by the intensity of the transmitted beam. The aperture of the incident beam is always finite, and will have a definite area A. Consequently, this will produce a diffraction pattern of its own. If we take the amplitude of the incident wave as unity, the intensity in the exact forward direction would be A^2/λ^2, and the pattern would extend over a solid angle of the order of λ^2/A. We may therefore speak of two quantities as representing the intensity of the transmitted beam: (*a*) the intensity in the exact forward direction, and (*b*) the total intensity in the diffraction pattern, which is in general comprised within a small solid angle. If the beam is brought to a focus on a photographic plate and we take a microphotometer record of the intensity, then the height of the peak would represent the quantity (*a*). If, on the other hand, we focus the beam on a photocell, the current produced would represent the quantity (*b*). The distinction between the two is necessary, since the reduction due to the existence of an obstacle is not the same for both, as is shown below.

We will now consider the general method of finding the reduction in the forward intensity. Let U be the amplitude of the diffraction pattern produced by the aperture A. Then $I = \int |U|^2 d\Omega$, where Ω is the solid angle, represents the total energy in the diffraction pattern, the integration being done over all the solid angles where $|U|^2$ is finite. On the other hand, $I_0 = |U_0|^2$ represents the intensity in the forward direction per unit solid angle, if U_0 is the amplitude in this direction.

Now, suppose that an obstacle of area a be introduced in the path of the beam. Then, this would produce a diffraction pattern of its own. Let u be the amplitude in the diffraction pattern due to an aperture complementary to the obstacle. Then, the amplitude in the pattern due to the obstacle is, by Babinet's principle equal to $-u$. The two patterns due respectively to the aperture A, and the obstacle a will be superposed, the

second of which would obviously cover a larger solid angle. In this case, the total energy in the transmitted beam is

$$I' = \int |U - u|^2 \, d\Omega, \tag{16}$$

where the integration is to be carried out over the same range of solid angles, as was done for I. On the other hand, the intensity in the exact forward direction is

$$I_0' = |U_0 - u_0|^2. \tag{17}$$

From these formulæ, the reduction in intensity either of I or of I_0 can be calculated.

Coming now to the particular case in which we are interested, assume for simplicity that the obstacle α is spherical, and has a radius a. The diffraction would now consist of two parts, (1) the pattern due to the aperture A, covering a solid angle of the order of λ^2/A, and (2) the pattern due to the obstacle, covering a much larger solid angle of the order of λ^2/α.

Evidently, $I_0 = A^2/\lambda^2$, and $I_0' = (A - \alpha)^2/\lambda^2$, so that the ratio $I_0'/I_0 = (1 - 2\alpha/A)$, since α is small. Thus, the intensity in the exact forward direction is reduced in the ratio $(1 - 2\pi a^2/A) : 1$, or double the reduction as given by the geometric theory.

In order to evaluate I', assume that the pattern due to aperture extends over a solid angle Ω_0; Ω_0 is $\simeq \lambda^2/A$. Then,

$$I = \int_{\Omega_0} |U|^2 \, d\Omega = A. \tag{18}$$

Take now the whole pattern due to the aperture and the obstacle, and assume it to extend over the solid angle Ω. Then the total energy in this is

$$\int_{\Omega} |U - u|^2 \, d\Omega. \tag{19}$$

But, outside Ω_0, $U = 0$, so that the above expression can be written as

$$\int_{\Omega_0} |U - u|^2 \, d\Omega + \int_{\Omega - \Omega_0} |u|^2 \, d\Omega. \tag{20}$$

Hence,

$$I' = \int_{\Omega_0} |U - u|^2 d\Omega = \int_{\Omega} |U - u|^2 \, d\Omega - \int_{\Omega - \Omega_0} |u|^2 \, d\Omega.$$

Now, Ω_0 is $\ll \Omega$, so that the energy of the pattern due to the obstacle contained within Ω is very negligible. Hence, $\int_{\Omega - \Omega} |u|^2 \, d\Omega = \int_{\Omega} |u|^2 \, d\Omega$, which must be equal to α, by Babinet's principle. Also, $\int_{\Omega} |U - u|^2 \, d\Omega$ will be equal to $(A - \alpha)$, so that

$$I' = A - 2\alpha.$$

Thus, $$I'/I = (1 - 2\alpha/A), \quad (21)$$

i.e., the total intensity is also reduced in the same ratio as the forward intensity, the reduction being double that given by the geometric theory.

A physical interpretation can also be given for this double loss in the total transmitted intensity. The cause has already been explained as due to the effect of diffraction. Now, by Babinet's principle, the energy in the diffraction pattern due to an opaque disk is the same as that in the pattern due to an aperture of the same size, and hence equal to α. In this way, the forward wave loses an amount of energy α due to diffraction, in addition to an equal amount lost due to the blocking by the opaque obstacle. This is exactly what is shown by the formulæ above.

However, it must be noted that we have assumed in this derivation that the energy of the diffraction pattern due to α, contained in the solid angle embraced by the pattern of the transmitted light is negligible. This will be true, only if $\alpha \ll A$. However, if this is not the case, then the additional loss of energy will not be α, but less. In general, therefore

$$I'/I = (1 - c\alpha/A), \quad (22)$$

where $c = 2$, only if α is small compared to A. If the obstacle is of dimensions comparable to those of A, then the pattern due to it will be quite small, and we have to take c nearly equal to 1, as is found from geometrical considerations. This is quite natural, since geometrical optics applies rigorously when large sizes are taken into consideration. However, in the exact forward direction, the reduction in intensity is always double that given by the geometric theory. The double loss of energy, and the conditions determining its occurrence do not seem to have been pointed by anybody so far.

We will now consider what the effect of a cloud of particles would be according to wave theory. Assuming that the incident wave is represented by $u = \exp(2\pi ict/\lambda)$ at $x = 0$, which corresponds to the beginning of the cloud, the wave at a distance x within the medium can be represented by

$$u = e^{-kx} e^{\frac{2\pi i}{\lambda}(ct - \mu x)}$$

where μ is the refractive index, and k, the absorption coefficient. Taking now a thin layer of the medium of thickness dl, the number of particles in this thickness per unit area of cross-section will be $N dl$. As already seen, the effect of each particle would be to reduce the amplitude by α. Hence, the wave, after passing the thickness dl would be represented by

$$(1 - N\alpha dl)\, e^{\frac{2\pi i}{\lambda}(ct - dl)}.$$

Comparing this with the above equation, we get $\mu = 1$, and $\exp(-kdl) = (1 - N\alpha dl) = \exp(-N\alpha dl)$. Hence, the absorption coefficient for the amplitude is $k = N\alpha$, and that for intensity will be double this, namely $2N\alpha$. Hence, the intensity of the beam after passing through a length l of the medium is

$$I_l = I_i\, e^{-2N\alpha l} \tag{23}$$

where I_i is the incident intensity. For spherical particles of radius a, this gives

$$I_l/I_i = e^{-2\pi a^2 N l}. \tag{24}$$

This expression differs from that given by the geometric theory only in that there is an additional factor 2 in the exponent, the reason for which has already been explained.

4. *Passage of Light through Transparent Spheres*

This case is of practical importance in connection with the study of the transmission of light through a cloud of water droplets, as in fog, cloud or rain. Mallock has wrongly supposed tnat one can take the water particles as opaque. This is, *prima facie*, not true, since the droplets are by their very nature transparent, and not opaque. Consequently, portions of the wave-front transmitted through the droplets must also be taken into consideration. The portions passing through the droplet will, however, be retarded with respect to the rest of the wave-front, the extent of the retardation depending on the distance each portion has passed through the drop. Changes in amplitude will also occur in the wave-front on account of the convergence of the rays passing through the drop. These features must be taken into account in a proper theory of the phenomena.

The actually observed facts also show that the drops cannot be regarded as opaque. The work of Barus (1907, 1908), and of others has shown that the transmitted light is often highly coloured and that this colour is complementary to the colour of the light scattered by the particles themselves in the forward direction. Also, these colours are found to undergo periodic changes with increase in particle size, reappearing for values of the radii in the ratio of natural numbers. These phenomena indicate that the light transmitted through the drop must be taken into account.

The most rigorous method of attacking the problem would be to use the electromagnetic theory of light, and to determine the effect of a dielectric sphere on the field. Such a theory has been given by Mie (1908), Debye (1909), Rayleigh (1910) and Bromwich (1920). However, this method does not give any simple expression for the intensity of the diffracted and the transmitted light, if the size of the droplets is not small compared with

the wave-length of light. The nature of the phenomena could only be studied by numerical computation, which again is prohibitively tedious. It is therefore desirable to give a theory based on optical principles, which would yield expressions that can readily be applied.

We shall now attempt to give such a theory for the diffraction of light by a transparent sphere. The theory will be developed to include also the diffraction of light in directions inclined to the forward one. As already said, one must take into account both the amplitude and phase changes that occur in the transmitted wave-front. However, if we assume that the difference in refractive index between the sphere and the surrounding medium is small, then we can as a good approximation, take that the effects of refraction are negligible. In this case, we can neglect the changes in amplitude, and take only the phase-changes into consideration, the path retardation of any portion of the wave-front being calculated on the assumption that it has passed straight through the drop.

T. A. S. Balakrishnan (1941) has developed a theory for the diffraction by a single droplet based on similar ideas. In this paper, the theory is worked out more fully, and it is extended to the consideration of the transmission and forward scattering by a cloud of transparent spherical droplets.

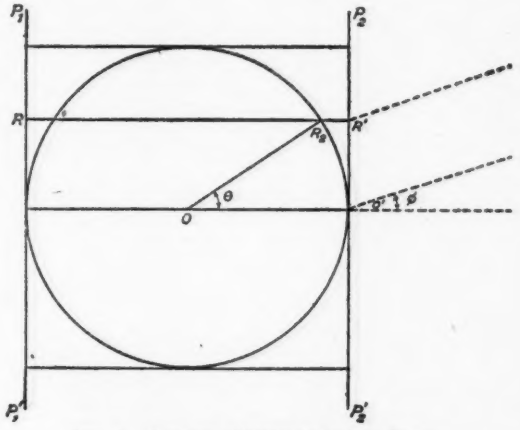

FIG. 1. Passage of Light through a Sphere

5. *Derivation of the Formulæ*

In this section, the formulæ for the diffraction of light by a spherical droplet is derived, for the case when the amplitude changes can be neglected, and the wave-front can be supposed to have passed straight through the

drop. Let P_1RP_1' be a plane wave-front which, after passing through a droplet of centre O, and radius a, emerges out as $P_2R'P_2'$ (Fig. 1). The phase of the emergent wave-front will evidently not be uniform throughout, but will lag behind in the portions which have passed through the drop. Let RR_2R' be any ray, and let the angle between OR_2 and OO' be θ. Then, this ray suffers a retardation of $2a(\mu-1)\cos\theta$ with respect to the portions of the wave-front passing outside the drop. Thus, the plane wave-front, on the whole, would be distorted into one having a concave part in the centre.

Our idea is to find the Fraunhofer diffraction pattern of this emergent wave-front, *i.e.*, we wish to find the intensity in a direction making an angle ϕ with the incident direction. For this, we proceed in a manner analogous to the usual theory of diffraction by a circular aperture (*vide* Max Born, *Optik*, pp. 155 *et seq.*). Taking polar co-ordinates, if (r, α) be the co-ordinates of an element at R′ with respect to O′ as origin, and the horizontal radius as the initial position of the radius vector, then the diffracted amplitude in the direction ϕ due to this element is

$$\frac{1}{\lambda}\sin\frac{2\pi}{\lambda}\{ct-2(\mu-1)a\cos\theta+r\cos\alpha\sin\phi\}\,r\,dr\,d\alpha,$$

where the amplitude of the incident wave-front is assumed to be unity, and λ is the wave-length of the light. Substituting $a\sin\theta$ for r, and integrating between the proper limits, we get that the amplitude in the direction ϕ due to the portion of the wave-front which has passed through the drop is

$$X_1=\frac{a^2}{\lambda}\int_0^{\pi/2}\int_0^{\pi}\sin\frac{2\pi}{\lambda}\{ct-2(\mu-1)a\cos\theta+a\sin\theta\cos\alpha\sin\phi\}\sin\theta\cos\theta\,d\theta\,d\alpha.$$

Putting $\chi=2\pi ct/\lambda$, $\xi=4\pi(\mu-1)a/\lambda$, and $\eta=2\pi\sin\phi\,a/\lambda$,

$$X_1=\frac{a^2}{\lambda}\int_0^{\pi/2}\int_0^{2\pi}\sin(\chi-\xi\cos\theta+\eta\sin\theta\cos\alpha)\sin\theta\cos\theta\,d\theta\,d\alpha.$$

Integrating with respect to α, this can be written as

$$X_1=\frac{2\pi a^2}{\lambda}\int_0^{\pi/2}J_0(\eta\sin\theta)\sin(\chi-\xi\cos\theta)\sin\theta\cos\theta\,d\theta. \qquad (25)$$

The effect of the portion of the original undisturbed wave-front covered by the drop is obtained by substituting $\mu=1$ in the above expression,

which makes $\xi = 0$, and gives the value

$$X_2 = \frac{2\pi a^2 \sin \chi}{\lambda} \int_0^{\pi/2} J_0(\eta \sin \theta) \sin \theta \cos \theta \, d\theta$$

$$= \frac{2\pi a^2}{\lambda} \cdot \frac{J_1(\eta)}{\eta} \cdot \sin \chi. \tag{26}$$

The amplitude X of the diffracted wave due to the complete wave-front which has traversed the drop is the sum of the amplitude due to the portion which has passed through the drop, and that due to the portion which has passed outside it. The former is given by X_1, and the latter by subtracting X_2 from the effect due to the undisturbed wave-front, which we may call X_0. In the region outside that covered by the source, X_0 is evidently zero, so that

$$X = X_1 - X_2. \tag{27}$$

In the forward direction, *i.e.*, for $\phi = 0$, X_0 is finite, so that the resultant amplitude, X′, in the forward direction, is

$$X' = X_0 + X_1' - X_2', \tag{28}$$

where X_1' and X_2' are the values of X_1 and X_2, when $\phi = 0$. For this particular case, since $\phi = 0$, η vanishes, and the integral in the expression for X_1 can be easily integrated, giving

$$X_1' = -\frac{2\pi a^2}{\lambda \xi^2} \int_0^{\pi/2} \sin(\chi - \xi \cos \theta)(\xi \cos \theta) \, d(\xi \cos \theta)$$

$$= \frac{2\pi a^2}{\lambda \xi^2} \sin \chi \, [\xi \sin \xi + \cos \xi - 1] + \frac{2\pi a^2}{\lambda \xi^2} \cos \chi \, [\xi \cos \xi - \sin \xi].$$

Also, $X_2' = \frac{\pi a^2}{\lambda} \sin \chi$, since $\underset{\eta \to 0}{\text{Lt}} \frac{J_1(\eta)}{\eta} = \frac{1}{2}$, so that

$$X' = X_0 + 2\pi a^2/\lambda \cdot \sin \chi \cdot (\cos \xi/\xi^2 + \sin \xi/\xi - 1/2 - 1/\xi^2)$$
$$+ 2\pi a^2/\lambda \cdot \cos \chi \, (\cos \xi/\xi - \sin \xi/\xi^2).$$

Now, if A be the aperture of the beam, then X_0 is clearly equal to $A/\lambda \cdot \sin \chi$, so that

$$X' = \frac{A \sin \chi}{\lambda} \left[1 + \frac{2\pi a^2}{A} \left\{\frac{\cos \xi}{\xi^2} + \frac{\sin \xi}{\xi} - \frac{1}{2} - \frac{1}{\xi^2}\right\}\right]$$
$$+ \frac{A \cos \chi}{\lambda} \left[\frac{2\pi a^2}{A} \left\{\frac{\cos \xi}{\xi} - \frac{\sin \xi}{\xi^2}\right\}\right]. \tag{29}$$

Now, the quantity $2\pi a^2/A$ is small, (since A can be made as large as we please), so that its square can be neglected. Hence, the intensity in the

forward direction due to the wave-front that has passed through a droplet is

$$I_0' = \frac{A^2}{\lambda^2}\left[1 - \frac{4\pi a^2}{A}\left\{\frac{1}{2} + \frac{1}{\xi^2} - \frac{\cos\xi}{\xi^2} - \frac{\sin\xi}{\xi}\right\}\right] = \frac{A^2}{\lambda^2}\left[1 - \frac{4\pi a^2}{A} f(\xi)\right] \text{ (say).}$$

This expression gives the reduction in intensity due to a single droplet. In order to find the effect of a cloud of droplets, we proceed in a manner similar to what we adopted for a cloud of opaque particles. Let the incident wave be represented by $u = \sin\left[\frac{2\pi}{\lambda}(ct - \mu x)\right]$, the origin being at the boundary of the cloud. Then, from (29), the amplitude of the wave after passing through a layer of the cloud of thickness dl is

$$u' = [1 - N dl\, 2\pi a^2 f(\xi)] \sin\frac{2\pi}{\lambda}(ct - dl) - N dl\, 2\pi a^2 g(\xi) \cos\frac{2\pi}{\lambda}(ct - dl), \quad (30)$$

where $g(\xi)$ is the expression $(\sin\xi/\xi^2 - \cos\xi/\xi)$. From the wave-theory, we know that u' must be of the form

$$u' = e^{-kdl} \sin\frac{2\pi}{\lambda}(ct - \mu dl) \quad (31)$$

Putting the expression (30) above in this form

$$u' = [\{1 - 2\pi N a^2 f(\xi)\, dl\}^2 + \{2\pi N a^2 g(\xi)\, dl\}^2]^{1/2} \sin\left\{\frac{2\pi}{\lambda}(ct - dl) - \epsilon\right\},$$

where $\tan\epsilon = 2\pi N a^2 g(\xi)\, dl/\{1 - 2\pi N a^2 f(\xi)\, dl\}$.

Since both the quantities $2\pi N a^2 f(\xi)\, dl$ and $2\pi N a^2 g(\xi)\, dl$ are small compared to unity, this may be written as

$$u' = [1 - 2\pi N a^2 f(\xi)\, dl] \sin\{2\pi(ct - dl) - \epsilon\}$$

$$= e^{-2\pi N a^2 f(\xi) dl} \sin\left\{\frac{2\pi}{\lambda}(ct - dl) - \epsilon\right\},$$

where $\tan\epsilon = 2\pi N a^2 g(\xi) dl = \epsilon$.

Comparing this with (31), we see that $k = 2\pi N a^2 f(\xi)$ and $\mu = 1 + N\lambda a^2 g(\xi)$. Since the absorption coefficient for intensity is double that for amplitude, we get that the intensity after passing through a length l of the cloud is

$$I_l = I_0 \exp[-4\pi a^2 N l f(\xi)], \quad (32)$$

where I_0 is the incident intensity.

6. *Colours of the Transmitted Beam*

Equation (32) above gives the formula for the attenuation of the directly transmitted beam. Ordinarily, for particles whose radii are larger than the wavelength of light, ξ is large, so that terms in $1/\xi^2$ can be neglected. (32) then goes over into the form

$$I_l/I_0 = \exp[-4\pi a^2 N l\,(1/2 - \sin\xi/\xi)]. \quad (33)$$

This expression clearly shows that the attenuation of the transmitted light is different for different wavelengths, the attenuation coefficient depending on ξ, and hence on λ. Taking particles of a fixed size, the value of ξ will evidently depend on the wavelength λ of the light, and from (33) it is clear that the intensity will be a maximum for those for which sin ξ is $+1$, and a minimum for which sin ξ is -1. Now, the visible spectrum comprises the range of values $0{\cdot}4\,\mu$ to $0{\cdot}7\,\mu$ of λ, so that if the drops are small enough, there will be only one maximum, and the colour of the source will be due to the predominant effect of the range of wave-lengths in the neighbourhood of this maximum. This explains the reason why the transmitted light is vividly coloured. In Fig. 2, $f(\xi)$ is plotted against ξ, and it shows the manner in which $f(\xi)$ undergoes maxima and minima.

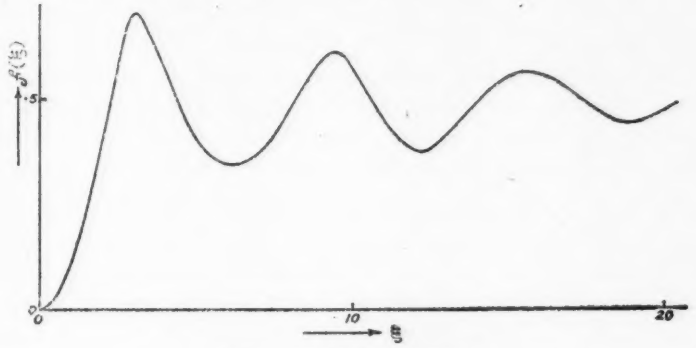

FIG. 2. Graph of $f(\xi)$ against ξ

As the size of the droplet is changed, the value of λ for which the maximum occurs will vary, and consequently, the colour of the transmitted beam will also vary. Since the varying function is sin ξ, we should expect a cyclic change of colours as ξ increases; the colours will repeat themselves for values of ξ which are in the ratio of natural numbers, *i.e.*, for values of the radius a which are in this ratio. This is in accordance with Barus' observations (1907).

If the size of the droplets is further increased, the range of values of ξ may comprise two or more maxima, and also the difference between the maxima and minima of intensity will not be pronounced, since $(\sin \xi)/\xi$ becomes intrinsically small. Hence the colours will become less and less vivid. In the limiting case of very large values of ξ, $\sin \xi/\xi$ becomes small compared to $\frac{1}{2}$, and (33) reduces to

$$I_l/I_0 = \exp[-2\pi a^2 N l]. \qquad (34)$$

There is no term depending on λ, so that no colours will be exhibited. It is also interesting to notice that this is identical with the intensity in the forward direction given by a large opaque sphere of the same size. The following simple physical argument shows that this must be the case. When the particles are large, the path retardation between the central ray, and the rays passing outside the drops will comprise a number of wavelengths. We can therefore divide the portion of the wave-front which has passed through the droplets into half-period zones, of which the outer ones will practically cancel out. The resultant effect will only be due to the innermost rings, the area of which relative to the total area of the wave-front that has passed through the drop is smaller, the larger the drop. Thus, the forward intensity must approach the value for an opaque disc. In fact, the vanishing of the term $(\sin \xi)/\xi$ mathematically represents this fact.

7. *The Intensity of Forward Scattering*

As already mentioned, we can employ our theory to consider the nature of the light scattered in the forward direction by the droplets. This would comprise the region just outside the region covered by the source. Here, ϕ is small, but the effect of the incident wave is zero. The amplitude is therefore $X = X_1' - X_2'$. Substituting for X'_1 and X'_2,

$$X = \frac{2\pi a^2}{\lambda} \sin \chi \left[\frac{\cos \xi}{\xi^2} + \frac{\sin \xi}{\xi} - \frac{1}{2} - \frac{1}{\xi^2} \right] + \frac{2\pi a^2}{\lambda} \cos \chi \left[\frac{\cos \xi}{\xi} - \frac{\sin \xi}{\xi^2} \right]. \quad (35)$$

As before, neglecting terms in $(1/\xi^2)$ as an approximation, we get the intensity as equal to

$$(2\pi a^2/\lambda)^2 \, (1/4 - \sin \xi/\xi). \quad (36)$$

This expression refers to the intensity diffracted by a single droplet. But the light reaching it, and the light diffracted have both to travel through the fog, and the total path will not differ from l, since ϕ is small. Hence, the expression (36) has to be multiplied by the expression

$$\exp \, [- 4\pi a^2 \mathrm{N} l \, (1/2 - \sin \xi/\xi)].$$

Also, since we have assumed a random distribution of the droplets, there will be no phase correlation between the light scattered by different particles. Hence, the light diffracted out of the cloud in the forward direction, per c.c., is $I_f = \mathrm{N} \, (2\pi a^2/\lambda)^2 \cdot \, (1/4 - \sin \xi/\xi)$.

$$\exp \, [- 4\pi a^2 \mathrm{N} l \, (1/2 - \sin \xi/\xi)]. \quad (37)$$

This is of the form Bxe^{-Cx}, where $x = \mathrm{N} \, (1/4 - \sin \xi/\xi)$. The corresponding expression for the transmitted light is $I_t = I_0 e^{-Cx}$. Now, the former expression is zero when $x = 0$, and reaches a maximum for $x = 1/\mathrm{C}$, after which it diminishes. The latter is a maximum at $x = 0$, and steadily

diminishes as x increases. Hence, if the maximum value of x, upon which the colours depend, is less then 1/C, then corresponding to it I_f will be a maximum, while I_l is a minimum. Similarly, for the minimum value of x, the reverse will be the case. Thus, the colour in the region immediately surrounding the source will be complementary to that of the source itself.

However, it is to be noted that for this phenomenon to be prominent, the maximum value of x must be less than 1/C, or Nl must be less than a quantity which depends on the size of the droplets. In other words, if the density of the fog is fixed, there is a limit to the depth of the fog beyond which the colours are not complementary, but will tend to be more and more identical. The phenomenon of complementary colours can best be observed with fairly thin columns of cloud, as for instance when the sun is seen through the puffs of steam emitted by a locomotive.

In the limiting case of very large drops, the expression (36) for the forward intensity reduces to $(\pi a^2/\lambda)^2$. This is again identical with that for an opaque obstacle of the same size.

8. *Concluding Remarks*

The theory developed in this paper for the transmission of light through a medium containing a cloud of particles cannot be expected to be strictly valid for very small sizes of the particles, since Huygens' principle, and the idea of complete wave-fronts, on which it is based cannot be applied when very small dimensions are concerned. The correct procedure in these cases is to apply the theory of scattering of light, as for example, the theory developed by Mie and others (*loc. cit.*). These however do not yield any simple expressions; but for very small droplets, the theory of Rayleigh can be used. In this case, the attenuation has actually been worked out by Lord Rayleigh (*loc. cit.*) and can be represented by

$$I_l/I_0 = \exp\left[-\mathrm{N}l\,\frac{128\pi^5}{3}\cdot\frac{(\mu^2-1)^2}{(\mu^2+2)^2}\cdot\frac{a^6}{\lambda^4}\right]. \tag{38}$$

In the case when $(\mu-1)$ is small, $(\mu^2+2)^2$ can be put as equal to 9, so that

$$I_l/I_0 = \exp\left[-128\pi^5(\mu^2-1)^2\,a^6\mathrm{N}l/27\lambda^4\right]. \tag{39}$$

This theory is certainly not true for larger particles. It is not possible to say exactly at what stage the line of argument advanced in this paper cases to be valid, although we can take it to be applicable, if the radius is larger than the wave-length of light. This point will be discussed more in detail in a later publication.

I wish to express my sincere thanks to Prof. Sir C. V. Raman for suggesting the problem, and for the kind and encouraging guidance he gave during the course of the investigation.

Summary

The phenomenon of the transmission of light through a cloud of particles distributed at random in a transparent medium is theoretically investigated on the basis of wave-optics. The cases of both transparent and of opaque particles are considered, and it is found that the transmitted beam is progressively attenuated, the intensity diminishing exponentially with the increase in the thickness of the medium. The actual values of the attenuation coefficient are calculated in both cases. It is found that for an opaque particle, the diminution of intensity in the forward direction is actually double what would be expected from simple geometric considerations. For transparent particles, the transmitted intensity shows spectral variations, and this explains some of the phenomena hitherto not well understood, such as the colours shown by the transmitted light, and the complementary nature of this colour and the colour of the light scattered by the cloud in the forward direction.

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TREND ANALYSER

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1. Introduction

THE Analysis of Time Series is of great importance in economic and other social studies. The eye of the economist always rests upon the Trend of prices and closely follows its rise and fall. The sales manager takes keen interest to know how the volume of sales is changing, when and why it fluctuates. Thus the plans of business and schemes of economic forecasting are all based on the study of Trends.

Several methods appropriate to the Analysis of Time Series have been developed. A straightforward and easy procedure is the graphical method. The graph may show a general trend but it lacks in precision and accuracy. Hence we resort to more refined methods of analysis. From the practical point of view the different methods of getting the trend may be looked upon as methods of graduation or smoothing. These methods may be classified under the following heads :—

1. Method of Moving Averages.
2. Method of Differences.
3. Method of Least Squares.
4. Method of Orthogonal Polynomials.

We discuss these methods with the help of a numerical example and finally give a novel method of getting the trend by a mechanical device.

2. Methods of Graduation

Column 2 of Table I gives the figures of Bank Clearings in New York City for 41 years from 1860 to 1900.[1] These figures have been graduated by the different methods and the results are given in the remaining columns.

In the Method of Moving Averages the average value for a consecutive number of years is taken as the normal or trend value for the middle of the

[1] Frederic C. Mills. Table 59, p. 259.

period covered in the calculation of the average. Columns 3 and 4 give the results of the five-year and nine-year moving averages. Fig. 1 shows the trend as determined by the Method of Moving Averages. The nine-year average curve is smoother than the five-year one. Either of these gives more or less an idea of the general trend.

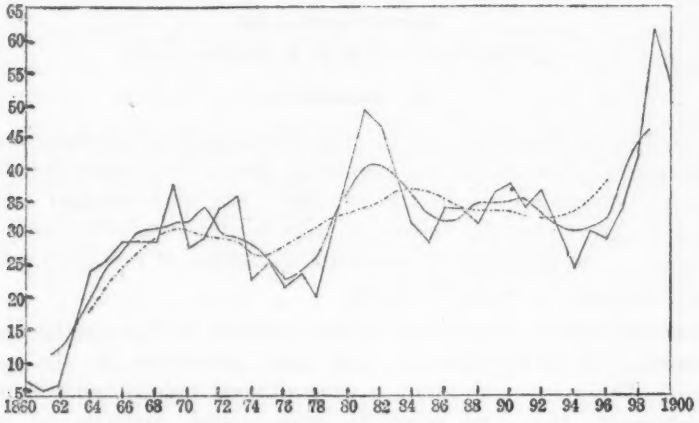

FIG. 1

——— Observed values (Bank clearings).

5-year moving average.

-·-·-·- 9-year moving average.

It is however desirable to get the trend based on a more mathematical basis.[2] Attempts in this direction have given rise to several methods which involve the successive differences of the observed values. We shall consider only one of these—the Summation formula of Spencer—namely,

$$v_0 = [5] \cdot [5] \cdot [7] \cdot (1 - 4\delta^2) \cdot u_0 \div 125.$$

The graduated values by this formula are given in column 5.

The methods of graduation based on differences have been studied in detail by Whittaker and Robinson.[3] They offer the following remarks regarding these.—"The methods described above fulfil their purpose of smoothing out irregularities from observational data, in a way which is on

[2] The one fundamental difference between moving averages as a measure of trend and the mathematical curve is that the one implies no definite law and is a flexible measure of trend adopting itself to changing conditions, purporting to be nothing more than an empirical approximation to the drift of the series, while the other assumes a law of change underlying the variations and postulates a uniform and consistent trend capable of mathematical expression.

[3] Whittaker and Robinson, p. 303.

the whole efficient. From purely theoretical standpoint they are not altogether satisfactory since each of them contains arbitrary or empirical elements whose introduction does not appear logically necessary." These remarks are true of Spencer's formula. Another difficulty about the method is that we get only a small number of graduated values after all the labour in numerical computation.

We may now proceed to discuss the method of Least Squares which, of course, rests on a logical basis. This method may be briefly stated thus : "Supposing the ungraduated values u_x to be plotted as points against the corresponding values of x, fit a parabolic curve of some assigned degree j to the points $(u_{-n}, u_{-n+1}, u_0, \ldots u_n)$ determining the constants of the curve by the method of Least Squares and take the ordinate of this curve $x = 0$ as the graduated value of u_0."[4] If j be equal to $2k$ or $2k + 1$, we can fit a straight line to the data taking $k = 0$. Putting $k = 1$, we can fit a parabola of degree 2 or 3, and similarly for other values of k. To graduate the data at hand, we take $n = 10$, $k = 2$. The graduated values are given in column 6. The numerical work involved in this method is not anything less, if not more, than in the method of Spencer. We see also that the end values could not be graduated by this method.

In the methods discussed above, the trend is not given as a mathematical function. It is possible to derive a mathematical equation to represent the trend by assuming that it is given by

$$Y = a + bx + cx^2 + dx^3 + \ldots.$$

It is however advantageous to take the equation in the form

$$Y = A + B\mu_1 + C\mu_2 + D\mu_3 + \ldots.$$

where $\mu_1, \mu_2, \ldots.$ are a set of orthogonal polynomials. This modification has been systematised for numerical work by Fisher.[5] The several approximating curves are obtained in successive stages, the success of fitting terms of higher and higher degree being tested for significance at each stage.

The data given in Table I are graduated by this method and the graduated values for the Third and Fourth degree parabolas are given in columns 7 and 8.

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[4] Whittaker and Robinson, p. 291.

[5] Fisher, R. A., p. 140.

TABLE I

Bank Clearings of New York City and Graduated Values

Year	Obs. Val.	Moving Aver.		Spen.	L. Sq.	Orth. Poly.		Equ.	Tre.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1860	7·2	..	..	..	..	5·51	10·49	..	5·20
1861	5·9	..	..	..	..	9·58	12·78	..	6·87
1862	6·9	11·80	..	..	..	13·25	15·04	..	9·82
1863	14·9	15·56	..	..	..	16·54	17·26	..	15·71
1864	24·1	20·12	18·99	..	..	19·45	19·40	..	21·11
1865	26·0	24·48	22·34	..	..	22·02	21·45	..	24·83
1866	28·7	27·20	24·78	..	..	24·26	23·39	..	27·00
1867	28·7	29·86	27·27	..	..	26·19	25·20	..	28·47
1868	28·5	30·22	29·37	..	..	27·83	26·87	..	29·45
1869	37·4	31·34	30·63	..	..	29·21	28·39	..	29·55
1870	27·8	31·36	30·29	31·96	33·51	30·34	29·74	..	29·65
1871	29·3	33·76	29·89	31·23	32·17	31·24	30·92	..	29·55
1872	33·8	29·86	29·10	29·85	29·37	31·93	31·93	..	29·45
1873	35·5	29·32	28·52	28·11	26·55	32·44	32·76	..	28·47
1874	22·9	27·78	26·58	26·51	24·80	32·79	33·43	24·87	26·51
1875	25·1	25·68	26·73	25·47	24·19	32·98	33·92	25·15	25·53
1876	21·6	22·56	27·77	25·49	25·43	33·05	34·24	25·47	25·23
1877	23·3	23·82	29·50	26·72	27·14	33·01	34·41	26·42	25·53
1878	19·9	26·52	30·77	29·06	29·33	32·89	34·44	27·91	27·29
1879	29·2	32·08	32·38	31·95	32·46	32·70	34·33	30·72	31·42
1880	38·6	36·80	33·03	34·75	34·36	32·46	34·11	33·77	37·80
1881	49·4	40·30	33·77	36·82	36·23	32·19	33·79	35·98	44·18
1882	46·9	40·66	34·92	37·83	38·05	31·93	33·39	35·97	43·10
1883	37·4	38·58	36·42	37·71	38·35	31·67	32·95	34·14	39·27
1884	31·0	35·44	36·63	36·74	37·02	31·45	32·47	31·74	35·25
1885	28·2	32·74	36·33	35·43	36·16	31·28	32·00	29·47	33·38
1886	33·7	31·48	35·00	34·29	35·12	31·19	31·56	27·44	33·18
1887	33·4	32·46	33·53	33·64	33·91	31·19	31·19	24·34	32·89
1888	31·1	34·30	33·46	33·44	32·84	31·31	30·92	18·18	33·09
1889	35·9	34·30	33·48	33·37	33·25	31·56	30·79	..	33·18
1890	37·4	34·96	33·06	33·10	32·47	31·97	30·85	..	32·89
1891	33·7	34·98	32·63	..	..	32·55	31·13	..	32·40
1892	36·7	32·68	32·12	..	..	33·33	31·68	..	31·71
1893	31·2	31·18	32·38	..	..	34·32	32·55	..	30·44
1894	24·4	30·20	33·06	..	..	35·55	33·79	..	29·45
1895	29·9	29·54	35·66	..	..	37·03	35·46	..	30·04
1896	28·8	31·70	37·77	..	..	38·78	37·61	..	31·32
1897	33·4	38·98	..	..	..	40·84	40·29	..	34·85
1898	42·0	43·54	..	..	..	43·21	43·58	..	42·71
1899	60·8	..	..	..	..	45·91	47·53	..	57·44
1900	52·7	..	..	..	..	48·97	52·21	..	57·44

3. *The Equilibrium Method of Schumann*

T. E. W. Schumann has recently devised a new method of smoothing which he calls the equilibrium method.[6] His method is as follows :— Imagine an inextensible string tied to the points u_1 and u_n (the crude data being $y_1, y_2, \ldots . y_n$) the corresponding graduated values of y_1 and y_n. Initially the string is not stretched tightly but is supposed to exhibit a

[6] T. E. W. Schumann, *Phil. Mag.*, **26**, 179, pp. 970–83.

certain degree of slackness. Imagine springs to be attached at the points y_1 to y_n and to the string, each spring drawing the string in a positive or negative y direction with a force proportional to its own length. The equilibrium position of the string is the smooth curve for the data. By virtue of the equilibrium position Schumann equates the forces to zero and forms an equation which after some substitutions is brought to the form

$$u_r = \frac{p-a}{p+a}[\,y_r + (y_{r+1} + y_{r-1})\,a/p + (y_{r+2} + y_{r-2})\,a^2/p^2 + \ldots.],$$

where a lies between 0 and 0·5 and p is given by the equation $p^2 - p + a^2 = 0$. Schumann gives a few tables of coefficients to get the graduated values. Using the tables the data in Table I for 15 years from 1874 to 1888 has been graduated. The values obtained are given in column 9. ($a = 6/13$, $p = 9/13$.)

Schumann admits that his formula does not apply to the ends of the series. However 'the method exhibits some rather attractive features such as comparative ease of calculation, and the absence of any arbitrary manipulation to include the end terms in the graduation.' He also shows that close relationship exists between the method of Whittaker and the equilibrium method. 'It has an added interest in that it gives a clearer insight into what the smoothing process actually involves. In the second place it points the way towards the construction of a device whereby the smoothing of time series might be done mechanically without any lengthy calculations. It is not intended to describe such a mechanical device, but theoretically at least, it is possible to construct one.'

4. *The Mechanical Analyser*

The work of Schumann has suggested the possibility of constructing a mechanism that may be used to smooth a time series. This has made practical what Schumann feared would only remain theoretical. In principle this device consists of a number of smooth rings that may be adjusted to occupy positions corresponding to the points plotted from the data. This adjustment is effected by mounting each ring to a rod carrying a spring at either end. By this arrangement each ring can be made to take any desired position by stretching the springs and further about this position, it is capable of undergoing small oscillations along the line of the rod. A steel wire is now passed through the rings and the curve along which the wire comes to equilibrium gives the trend.

The mechanism[7] has been constructed in the Department of Mathematics of the University and is able to graduate 50 values spaced at equal

[7] This device has been constructed on the suggestion of U. S. Nair.

intervals (Fig. 2). It consists of a wooden frame 5′ × 3′ mounted vertically and from the top are suspended 50 springs each 10″ long at equal intervals of half an inch. To each spring is connected a brass rod 16″ long which

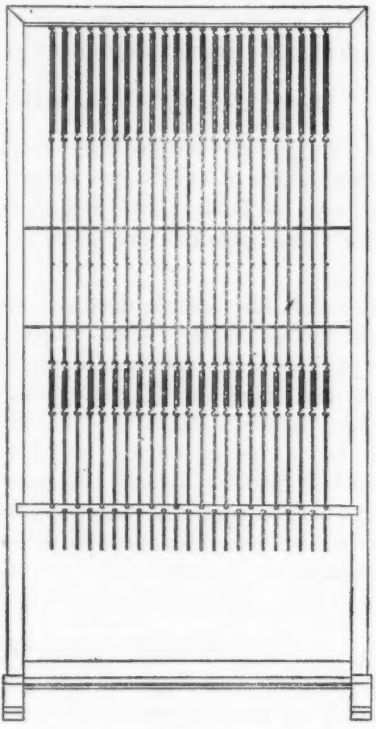

FIG. 2

moves vertically between holes drilled in two thin metal bars fixed horizontally to the frame 8″ apart. At the middle of each of these rods is mounted a small ring. The lower extremity of the rod is connected to another spring 5″ long and this spring carries a rod which may be pulled vertically and fixed in any position by means of screws. A movable platform to hold the graph paper on which is plotted the observed points fits into the space between the horizontal bars.

The working of the apparatus is simple. The observed values are plotted on a graph paper (scale $\frac{1}{2}$ inch as unit). The graph paper is fixed to the platform so that the points lie along the vertical rods. The rods are adjusted so that the rings take the positions of the points. A thin steel

wire is passed through the rings. The wire is then given a slight tension by pulling at both ends and allowed to come to the position of equilibrium. The new positions of the rings are plotted on the graph paper.

To make allowance for the elastic properties of the wire, we assume that the trend values are proportional to those as determined by the wire and calculate the constant by minimising the χ^2 obtained. Thus, if y_i and z_i give the observed and the corresponding wire values,

$$\chi^2 = \Sigma (y_i - a \cdot z_i)^2 / a z_i$$

Minimising this expression,

$$a^2 = \Sigma (y_i^2 / z_i) \div (\Sigma z_i).$$

Applying this method to data in Table I, the values in column 10 are obtained. Fig. 3 shows the graph of the trend as determined by the Trend Analyser.

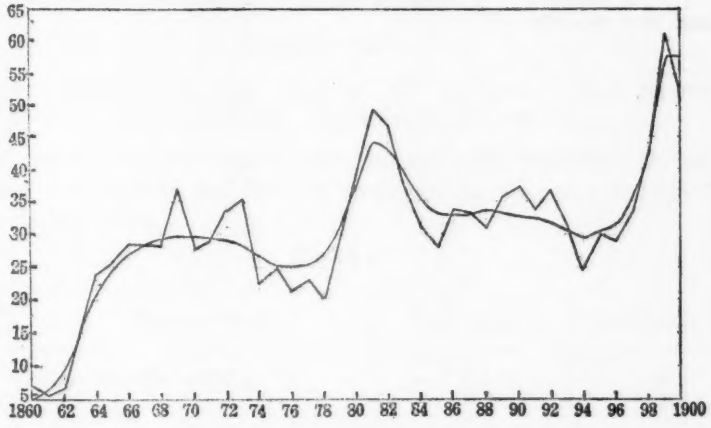

FIG. 3

——— Observed values (Bank clearings).

~~~ Values by Trend Analyser.

We may remark that it is possible to obtain any degree of smoothness with the Trend Analyser by using thicker and more inelastic wires. Thus, we are able to pass from the straight line to more and more complex curves to represent the trend. Also the actual curve can be drawn by passing the pencil point along the equilibrium position of the wire and finally shifting the axis of the argument through a distance corresponding to the value of a.
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3. Conclusion

In conclusion, the different methods of smoothing have been compared with the help of a numerical illustration and a novel method devised to determine the trend. The mechanical device has several advantages, the most important of these being its simplicity. Also without any lengthy calculations, it gives readily a series of curves having varying degrees of smoothness.

The author wishes to thank the University of Travancore for allowing him to carry on this work. He is also grateful to U. S. Nair for valuable advice he has given in the preparation of this paper.

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FIXED OIL FROM THE SEEDS OF *GNETUM SCANDENS* (ROXB.)

By N. S. Varier

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Gnetum scandens (Malayalam—*Odal*) is a lofty dioecious climber found commonly in the hilly tracts of the Western Ghats. The oil expressed from the seeds is used, in Travancore, for massage in cases of rheumatism, for lighting and, to a small extent, as an edible oil.

The seeds used in this work were obtained from North Travancore. They are oval and yellowish with a thick shell (70%) and a kernel (30%), which is firm and yellowish-brown in colour.

Experimental

Finely powdered kernels, dried at 105° (86 gm.), were successively extracted in a soxhlet apparatus with various solvents (Table I).

All the extracts were again extracted with light petrol (b.p. 50°–60° C.) and then with alcohol and thus resolved into an oil and a reddish-violet, non-crystalline solid.

TABLE I

Order of extraction	Name of solvent	Weight in grams	Percentage extracted
1	Light petrol (50°–60° C.)	12·3	14·2
2	Ether	2·8	3·4
3	Chloroform	3·7	4·3
4	Ethyl acetate	2·7	3·2
5	Alcohol	14·0	16·2
		35·5	41·3

The ash of the kernel contained carbonate, phosphate, sulphate, iron and potassium.

The oil had the following constants:—

Sp. gravity 30° C.	.. .. ..	0·9251
Ref. Index 30° C.	.. .. ..	1·4694
Acid value	.. .. ..	Nil

Saponification value	198·9
Iodine value (Winkler)	92·9
Hehner value	86·2%
Acetyl value	26·98
Unsaponfiable matter	0·81%

The mixed acids.—These were prepared by hydrolysing about 50 gm. of the oil with alcoholic potash and acidifying the soap so obtained, after removal of the unsaponifiable matter. A semi-solid mass (mean mol.wt.= 280·1), liquefying completely at 32° C., was obtained.

This was resolved into solid (70%) and liquid acids (30%) by Twitchells' lead-salt method.

The liquid acids had the following constants:—Mean mol. wt=281·4, Ref. index (30° C.)= 1·4637, and Iodine value= 110.

Bromination of liquid acids.—A well cooled solution of bromine in glacial acetic acid was added to an ethereal solution of the liquid acid (5·66 gm.) containing a little glacial acetic acid till a permanent colour remained. No insoluble hexabromide separated on keeping over ice. The ether was removed and the oily bromide was dissolved in hot light petrol (50°–60° C.) On cooling in ice, a brown powder (1·04 gm.) separated (m. pt. 112°–113° C.; after re-crystallisation from petrol it melted at 113°–114° C.). This is therefore linoleic acid tetrabromide. Bromine content (Stepanow's method) found =52·8%; calculated = 53%.

The filtrate was freed from light petrol, washed with dilute solution of sodium thiosulphate to remove any excess of bromine and then again dissolved in light petrol and dried. The petrol was removed and a liquid bromide resulted. This might be oleic dibromide.

Oxidation of liquid acids.—3 gms. of the liquid acids were dissolved in dilute alkali, the volume made up to 300 c.c. and a concentrated solution of 1·5 gm. of potassium permanganate solution added slowly, with continuous stirring, in the course of 15 minutes. The excess of permanganate was destroyed by passing sulphur dioxide. A white precipitate was formed. This was filtered, washed with a little ether to remove any free liquid acid and dried. The dried solid was extracted with cold ether to remove the tetra-hydroxy acid. The residue melted at 131°C. and had a molecular weight of 314·3. Hence it must be di-hydroxy-stearic acid. (Molecular wt. = 316.) The presence of oleic acid is thus confirmed.

The solid acids.—The solid acids (13 gm.) were separated into fractions by fractional crystallisation from alcohol and water.

TABLE II

Fraction	Weight	Melting point	Molecular weight
	gm.	°C.	grms.
1	5·0	68–69	283·5
2	4·11	62	281
3	1·49		
4 Residue	2·40		

Fraction 1 consists of stearic acid m. pt. = 68°–69° C.; Mixed melting point =68°–69° C.

Fraction 2 was separated fractionally into four parts.

TABLE III

Fractions	Weight	Melting point	Mean mol. weight
	gm.	°C.	gms.
A	0·811	69	282·7
B	0·362	68–69	280·3
C	1·697	35–36	
D Residue	1·240	30	

Fractions A and B (Table III) were confirmed to be stearic acid. Fractions C and D (Table III) were mixed with fractions 3 and 4 (Table II), and again resolved by the lead salt–ether method. The solid acids so prepared (about 4 gms.) were fractionally separated.

	Weight	Melting point	Molecular weight
	gm.	°C.	
X	0·95	59	264
Y	1·21	60	257
Z Residue	1·84	61·5	254

'X' is a mixture of palmitic and stearic acids. From table* it is a mixture of 10% stearic acid and 90% palmitic acid. Y gave a mixed melting point of 60° C. with palmitic acid and Z did not depress the melting point of palmitic acid.

Methyl esters of the solid acids.—The solid acids, prepared from a fresh quantity of the oil, were converted into methyl esters, in the usual manner with 3% methyl alcoholic hydrochloric acid. After drying in vacuum,

* *Allen's Commercial Analysis*, Vol. IV, p. 312.

20 gm. of the esters were distilled under reduced pressure (5 mm.) and five fractions were obtained.

All these fractions were saponified separately, the corresponding acids were liberated and these were fractionally crystallised to isolate the individual acids in the usual manner (Table V).

TABLE V

Fractions	B.P./5 mm.	Weight	Mean molecular weight	Melting point of the acid
	°C.	gm.		°C.
S_1	185 –190	0·5	254	60–61
S_2	190–195	3·0	258	61
S_3	195 –200	3·5	266	58–59
S_4	200 –210	8·0	283	67–68
S_5	210 –215	3·5	282	68
Residue (by difference) ..	..	2	..	67–68

The melting point of the acids after two crystallisations from dilute alcohol, are given in Table V. Those from S_1 and S_2 did not depress the melting point of pure palmitic acid and from S_4 and S_5, of pure stearic acid. S_3 is presumably a mixture of palmitic and stearic acid. The acids from the residue after four crystallisations gave a product melting at 67°–68° C. Thus only palmitic and stearic acids were present.

Unsaponifiable matter.—This was obtained as a waxy, brownish yellow mass. It gave the colour reactions for phytosterols.

Summary

The chemical and physical constants of the fatty oil from the seeds of *Gnetum scandens* (Roxb.) have been studied. The oil consists of the glycerides of oleic (27%), linoleic (3%), palmitic (14%) and stearic (56%) acids.

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CONTRIBUTIONS TO THE THEORY OF ORDERED GROUPS

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In this paper necessary conditions and sufficient conditions for that a given group can be ordered, will be investigated. It has been stated already in an earlier paper[1] of the author that in an ordered group the condition (*a*) : "every element $\neq 1$ is of infinite order" is satisfied and that every group in which holds (*b*) : "the commutatorgroup lies in the centre, and (*a*) is satisfied" can be ordered. It will be shown that the necessary condition (*a*) is not sufficient and that the sufficient condition (*b*) is not a necessary one. Moreover some more necessary conditions will be established.

§1. Let a, b (with and without indices) run over the integral numbers. Construct a group G of elements $\{a, b\}$ with the unitelement $\{0, 0\}$ by the following law of composition :

$$\{a_1, b_1\}\{a_2, b_2\} = \{a, b\},$$

where

$$a = a_1 + a_2,$$
$$b = (-1)^{a_2} b_1 + b_2.$$

Then G is a group since every element $\{a, b\}$ has an inverse element $\{-a, (-1)^{a+1} b\}$ and the associative law holds. The product of 3 elements is of course independent of the bracketing:

$$\{a_1, b_1\}\{a_2, b_2\}\{a_3, b_3\} = \{a_1 + a_2 + a_3, (-1)^{a_2+a_3} b_1 + (-1)^{a_3} b_2 + b_3\}.$$

Moreover for every positive integer m there is

$$\{a, b\}^m = \{ma, b\,[1 + (-1)^a + \ldots + (-1)^{(m-1)a}]\}$$

and the right hand side is equal to $\{0, 0\}$ if and only if $a = b = 0$. Hence condition (*a*) is satisfied in G. However

$$\{1, 0\}^{-1}\{0, 1\}\{1, 0\} = \{0, -1\} = \{0, 1\}^{-1}.$$

Suppose G is ordered and $\{0, 1\}$ a positive (negative) element, then its transformed elements are also positive (negative) whereas the inverse element

[1] "Ordered groups," *Proc. Ind. Acad. Sci.*, 1942, **16**, No. 4, Sec. A.

is negative (positive). Thus G cannot be ordered and therefore (a) is not a sufficient condition.

§ 2. Let R be a subring of an ordered field with the property that when a belongs to R, then $(a^2+a):2$ belongs to it. *E. g.*, R may be an ordered field, say a field of real numbers, or it may be the ring of the integral numbers. Let a, b, c, d, e run over the elements of R and construct a group H of elements

$$[a, b, c, d, e] \tag{1}$$

with the unitelement $[0, 0, 0, 0, 0]$ by the help of the following law of composition:

$$[a_1, b_1, c_1, d_1, e_1]\,[a_2, b_2, c_2, d_2, e_2] = [a, b, c, d, e], \text{ where}$$

$$a = a_1 + a_2, \quad b = b_1 + b_2, \quad c = c_1 + c_2 - a_2 b_1, \tag{2}$$

$$d = d_1 + d_2 + b_1 a_2 (a_2+1):2 - c_1 a_2, \quad e = e_1 + e_2 + b_1 a_2 b_2 + a_2 b_1 (b_1+1):2 - c_1 b_2$$

Then

$$[a,b,c,d,e]^{-1} = [-a, -b, -c-ab, -d-ba(a-1):2-ac, -e-ab(b-1):2-bc] \tag{3}$$

That the associative law holds, can be checked by an elementary calculation. Hence H is a group. Put

$$[1, 0, 0, 0, 0] = \alpha, \; [0, 1, 0, 0, 0] = \beta. \text{ Then the commutator}$$

$$(\alpha, \beta) = \alpha\beta\alpha^{-1}\beta^{-1} = [0, 0, 1, 0, 0] \text{ and}$$

$$\big(\alpha, (\alpha, \beta)\big) = [0, 0, 0, 1, 0], \; \big((\beta, (\alpha, \beta)\big) = [0, 0, 0, 0, 1] \; ;$$

thus (α, β) does not lie in the centre since it is not commutative with α and β. Therefore H does not satisfy the condition (b). However H can be ordered by the following rule : An element (1) is considered to be positive or negative according as the first of those of its five co-ordinates which are different from zero, is positive or negative. The unitelement is considered to be both positive and negative.

Now (2) shows that the product of two positive elements is positive and (3) shows that inverse elements have opposite signs. If one transforms (1) by any element of H, one gets an element of the form

$$[a, b, c+f_1, d+f_2, e+f_3],$$

where (as can be shown by an elementary calculation) $a = b = 0$ implies $f_1 = 0$ and $a = b = c = 0$ implies $f_2 = f_3 = 0$. Hence transformed elements have the same sign ; thus all the conditions for an ordered group are satisfied. Hence the sufficient condition (b) is not necessary.

If in particular the ring R is the ring of the integral numbers, H is generated by α and β. It may be mentioned without proof that in this case

H is isomorphic to $F/C_3(F)$ where F is a free group with 2 generators and C_3 denotes the third commutator group.

§ 3. It has been shown already in § 1 that a group in which an element $\neq 1$ is a transformed of its inverse element cannot be ordered. From this remark it follows immediately that if a manifold contains amphidromic curves which are not nullhomotopic, its fundamental group cannot be ordered. Of course the groupelement which corresponds to the amphidromic curve taken with a certain orientation is the transformed of the groupelement corresponding to the same curve taken with the opposite orientation, *i.e.*, to its inverse element; as the curve is not nullhomotopic the groupelement is different from 1. Hence the fundamental group of a one-sided surface cannot be ordered.

(2) Moreover in an ordered group an equation $x^m = c$ cannot have more than one solution for any positive m, since if a and b are two different solutions and $a > b$, then $a^m > b^m$. For $c = 1$ this condition is identical

(3) with the condition (a).

Another generalisation of condition (a) is the following theorem:

Let $a_1, \ldots, a_m$ *be* n *elements of an ordered group* A *and not all equal to* 1, *let* B *be the subgroup generated by them and* C(B) *its commutator group* ; *then* B/C(B) *contains elements of infinite order.*

Proof.—Suppose that the highest of the absolute values $|a_1|, \ldots, |a_n|$ is equal to a. Let A_1 be the group of the elements with absolute values atmost comparable with a, and A_0 the subgroup of the elements with absolute values infinitely small to a. Then $A_1 \supseteq B$ and a lies in B. In A_1 there exists a function[2] ${}_a lg$ which is zero for all the elements of A_0, but different from zero for the other elements of A_1. Now ${}_a lg\ (bc\, b^{-1} c^{-1}) = {}_a lg\, b + {}_a lg\, c - {}_a lg\, b - {}_a lg\, c = 0$. Hence the commutatorgroup of A_1 lies in A_0 and in particular $C(B) \subseteq A_0$. As ${}_a lg\, a^n = n$, none of the elements $a^{\pm 1}, a^{\pm 2}, \ldots\ldots$ is situated in $C(B)$. Hence the element $a\,C(B)$ of $B/C(B)$ is of infinite order.

[2] See *l.c.* [1] 5.

THE THEORY OF CORONÆ AND OF IRIDESCENT CLOUDS

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1. Introduction

Amongst the best-known phenomena of meteorological optics are the coronæ consisting of a central disc and one or more coloured concentric rings seen surrounding the sun or the moon, when viewed through thin clouds. The observed angular extension of the coronæ is variable, being generally between a degree and ten degrees. Closely related to these are the iridescent clouds exhibiting vivid colours, which are sometimes observed at distances of about 5° to 30° from the sun. The correlation between these two phenomena was recognized by Simpson (1912), who showed that iridescent clouds are nothing but parts of coronæ.

A mathematical theory of the corona was put forward by Airy, who assumed that the water droplets in the cloud act like opaque disks, and thus give rise to a diffraction pattern of bright and dark rings. Obviously, however, one is not justified in considering the water droplets as opaque, since actually they are transparent. Indeed, many of the experimental facts, as for example those observed by Barus (1907, 1908), Mecke (1920 *a*), and Mitra (1928) are not explained by the " opaque disk " theory. It is evident that the light transmitted through the droplets must be taken into account.

Mecke (1920 *b*) has tried to give a more complete theory, taking into account the rays that pass through the drops also. His method consists in dividing the effect of the droplet into three parts, due to diffraction, transmission and reflection respectively. The first part is taken to be the same as that due to an opaque disk of the same size. The transmission effect is calculated by tracing the refracted rays, and finding their divergence and phase change. A similar calculation is also made for the rays reflected from the surface of the droplet. All these parts are taken to be coherent, the amplitudes are added, and the resultant intensity is calculated.

It may be pointed out, however, that Mecke's application of the ideas of geometrical optics cannot be justified when the particles under consideration have a radius which is only a moderate multiple of the wave-length.

For, the transmitted rays themselves form a beam whose aperture is of the order of the diameter of the droplet, and hence comparable with the wave-length of light. Such a beam would give diffraction effects of its own, so that at the focal plane, where the geometric rays are supposed to come to a focus, the diffraction pattern would have a size about the same as that of the droplet itself. The ray-paths indicated by geometrical optics cease, therefore, to have any significance when one is dealing with droplets whose radius is only a small multiple of the wave-length. It is obvious, in these circumstances, that one must treat the problem on the basis of wave-optics alone, and not on the lines suggested by Mecke.

A theory of coronæ based exclusively on the ideas of the wave optics has been outlined by Balakrishnan (1941) taking into consideration the phase changes occurring in the passage of the light through the drops. The present author has used similar ideas in discussing the transmission of light through a cloud of transparent droplets, in a paper appearing earlier in these *Proceedings*. Both in that paper and in Balakrishnan's work, the aspect considered was the evaluation of the intensity in the forward direction, *viz.*, along the incident rays. The present paper concerns itself with the evaluation of the intensity in other directions, which is necessary to find the position and intensity of the rings observed in the corona.

2. *Derivation of the Formulæ*

It has been shown by the author (*loc. cit.*) that the amplitude of the wave diffracted by a droplet of radius a in a direction making an angle ϕ with the incident direction is

$$X = X_1 - X_2, \tag{1}$$

where

$$X_1 = \frac{2\pi a^2}{\lambda} \int_0^{\pi/2} J_0 (\eta \sin \theta) \sin (\chi - \xi \cos \theta) \sin \theta \cos \theta \, d\theta, \tag{2}$$

$$X_2 = \frac{2\pi a^2}{\lambda} \cdot \frac{J_1 (\eta)}{\eta} \cdot \sin \chi, \tag{3}$$

$$\xi = 4\pi (\mu - 1) a/\lambda, \text{ and } \eta = 2\pi a (\sin \phi)/\lambda, \tag{4}$$

the incident wave being represented by $\sin \chi$. The whole problem thus reduces to the evaluation of the integral in the expression for X_1. The integration is not simple for finite values of η. The author tried various methods of putting it in a form amenable for numerical computation. The best method was found to be one of partial integration, by which it could be put in the form of a series, of which the later terms are small compared with the first. The method consists in partially integrating $\sin (\chi - \xi \cos \theta)$,

keeping $J_0(\eta \sin\theta)$ as a constant. We may then write the integral as

$$-\frac{1}{\xi^2}\int_0^{\pi/2} J_0(\eta\sin\theta)\sin(\chi-\xi\cos\theta)(\xi\cos\theta)\,d(\xi\cos\theta),$$ which is

$$= \frac{1}{\xi^2}\Big[J_0(\eta\sin\theta)\{\sin(\chi-\xi\cos\theta)-\xi\cos\theta\cos(\chi-\xi\cos\theta)\}\Big]_0^{\pi/2} - \int \text{involving } J_1$$

$$= \frac{1}{\xi^2}\Big[J_0(\eta)\sin\chi - \sin(\chi-\xi)+\xi\cos(\chi-\xi)\Big] - \int \text{involving } J_1.$$

Normally, the value of ξ is fairly large, and we can neglect all terms invloving higher powers of $1/\xi$ as compared with those involving $1/\xi$. Then, the integral appearing in X_1 is $=\frac{1}{\xi}\cos(\chi-\xi)$, so that

$$X = K\,[\cos\chi\cos(\xi)/\xi + \sin\chi\{\sin(\xi)/\xi - J_1(\eta)/\eta\}] \tag{5}$$

where $K = 2\pi a^2/\lambda$. Hence, the intensity is

$$I = K^2\,[\{\cos(\xi)/\xi\}^2 + \{\sin(\xi)/\xi - J_1(\eta)/\eta\}^2]$$

$$= K^2\left[\frac{1}{\xi^2} + \frac{J_1^2(\eta)}{\eta^2} - \frac{2J_1(\eta)}{\eta}\cdot\frac{\sin\xi}{\xi}\right]. \tag{6}$$

This expression (6) is not very suitable when ξ is small, and also when η becomes comparable with ξ. In such a case, a second approximation can be worked out by expanding the integral for X_1 and summing up the most important terms (*vide* Appendix I) which gives

$$I = K^2\left[\frac{1}{\xi^2} + \frac{J_1^2(\eta)}{\eta^2} - 2\frac{J_1(\eta)}{\eta}\cdot\frac{\sin(\xi+\eta^2/2\xi)}{\xi}\right]. \tag{7}$$

A third approximation can also be obtained, which leads to the expression for I as

$$I = K^2\,[S^2 + \{C - J_1(\eta)/\eta\}^2],$$

where $$S = \frac{1}{\xi}\cos(\xi+\eta^2/2\xi) + \frac{1}{\xi^2}\left[\sin(\xi+\eta^2/2\xi) - \frac{\eta^2}{\xi}\cos(\xi+\eta^2/2\xi) - \frac{\eta^6}{8\xi^3}\sin(\xi+\eta^2/2\xi)\right]$$

and $$C = \frac{1}{\xi}\sin(\xi+\eta^2/2\xi) + \frac{1}{\xi^2}\left[\cos(\xi+\eta^2/2\xi) + \frac{\eta^2}{\xi}\sin(\xi+\eta^2/2\xi) - \frac{\eta^6}{8\xi^3}\cos(\xi+\eta^2/2\xi)\right].$$

3. *Discussion of the Expression for the Intensity*

For large values of ξ, the first approximation (6) is sufficiently accurate to evaluate the positions of the maxima. In order to determine how far it

is different from the more accurate expression (7), the author actually computed the intensity distribution given by both of them for a few cases. It was found that the higher degree of approximation did not appreciably alter the positions of the maxima and the minima, but that it gave values for the actual intensities at these positions different from those given by (6). Therefore the simpler expression (6) could be used for a discussion of the *positions* of the bright and dark rings in the corona, and, in the following discussion, this alone has been used.

We shall first derive a few general properties of the function on the right hand side of (6). For drops of a particular size, ξ is a constant, and the quantity which varies with the angle of diffraction ϕ is η. Hence, the varying part of I is given by

$$I' = K^2 [J_1^2(\eta)/\eta^2 - 2J_1(\eta)/\eta \cdot \sin(\xi)/\xi] \tag{8}$$

Putting $\sin(\xi)/\xi$ as k, this can be written as

$$I' = K^2 [J_1^2(\eta)/\eta^2 - 2kJ_1(\eta)/\eta] \tag{9}$$

Differentiating with respect to η,

$$\frac{dI'}{d\eta} = 2K^2 [J_1(\eta)/\eta - k] \cdot J_2(\eta)/\eta. \tag{10}$$

Hence, the maxima and minima of I′ occur for the values of η given by

$$J_1(\eta)/\eta = k \qquad (a)$$

$$J_2(\eta)/\eta = 0 \qquad (b)$$

Call the roots of the former $a_1, a_2, \ldots$ and those of the later $b_1, b_2, \ldots$.

Differentiating (16) once again, we get

$$[d^2I'/d\eta^2]_{\eta=a_m} = +2K^2J_2^2(\eta)/\eta^2 \tag{11}$$

which is always positive. Hence, at $\eta = a_m$, I is always a minimum.

Again, $[d^2I'/d\eta^2]_{\eta=0} = -K^2(1-2k)\,[d/d\eta\,\{J_2(\eta)/\eta\}]_{\eta=0}$

$$= -K^2(1-2k)\,(a + ve \text{ quantity}) \tag{12}$$

This is positive or negative according as k is $>$ or $< 1/2$. Hence, at $\eta = 0$, I is a maximum or minimum according as $k \gtrless 1/2$.

The other roots of (*b*), namely $b_1, b_2, \ldots$ may correspond to a maximum or minimum. It is also easily seen that they correspond to the value of η for which $J_1(\eta)/\eta$ has a turning value.

4. *The Rings of the Corona*

We are now in a position to discuss the disposition of the rings in the corona, which correspond to the maxima and minima of intensity. In this discussion, we shall limit ourselves to the case when monochromatic light is

used. The nature of the ring system with white light can be readily deduced from this.

The diffraction in the forward direction and the colours exhibited by the central portion of the corona have already been discussed in the previous paper. It is now only necessary to consider the outer rings of the corona. First of all, it can be shown that the thickness or density of the fog has no influence on the position of the rings in the corona. The intensity of the light diffracted by a single droplet at an angle ϕ is given by (6). But the light reaching it, and the light diffracted have both to travel through the thickness of the fog. This total path will not differ appreciably from the total thickness l of the fog, if the angle ϕ is not very large. Hence, from the equation (32) of the paper already cited, the intensity of the light diffracted per unit volume of the fog in the direction ϕ is

$$I = NK^2\left[\frac{1}{\xi^2} + \frac{J_1^2(\eta)}{\eta^2} - \frac{2J_1(\eta)}{\eta}\cdot\frac{\sin\xi}{\xi}\right]\exp\left\{-4\pi a^2 Nl\left(\frac{1}{2} - \frac{\sin\xi}{\xi}\right)\right\}, \tag{13}$$

where N is the number of droplets per unit volume. If the size of the droplet is fixed, ξ is a constant, and the position of the ring system is determined only by the expression within the square brackets. Neither N, nor l enters this expression, so that the ring system must be uninfluenced by the thickness of the fog. This has already been verified by Barus (1912) and also by Mecke (1920 *a*).

Taking, therefore, the expression within the square brackets, we have already shown that the position of its maxima and minima are given by the

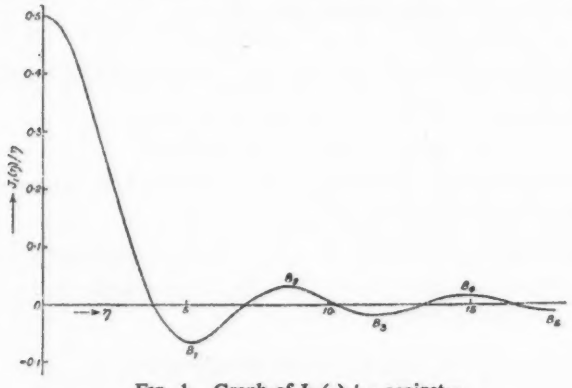

FIG. 1. Graph of $J_1(\eta)/\eta$ against η

roots of the equations (*a*) and (*b*). In the following discussion we shall limit ourselves to the first two or three rings, since, in general, the outer ones

would be faint. Also, from Appendix III, it will be seen that the series for C and S are not convergent if $\eta > \xi$, so that this discussion cannot be extended to values of $\eta > \xi$. This corresponds to $\sin\phi > 0{\cdot}667$, or $\phi > 41° 49'$. Since the coronæ usually observed do not extend beyond about 20°, this does not present any serious difficulty.

A careful analysis of the case shows that the existence and the number of the roots of (*a*) and (*b*) depend very much on the value of $k = \sin(\xi)/\xi$, and that there are four distinct cases. Denote by B_1, B_2,.... the maximum and minimum values of $J_1(\eta)/\eta$ which correspond to $\eta = b_1$, b_2, b_3, ... The course of the curve $J_1(\eta)/\eta$ is shown in Fig. 1, and the values of the B's and the *b*'s are tabulated in Table I below.

TABLE I

$b_1 = 5{\cdot}14$	$b_2 = 8{\cdot}42$	$b_3 = 11{\cdot}62$	$b_4 = 14{\cdot}78$	$b_5 = 17{\cdot}96$
$B_2 = -0{\cdot}0662$	$B_3 = +0{\cdot}0322$	$B_4 = -0{\cdot}200$	$B_5 = +0{\cdot}0140$	$B_6 = -0{\cdot}0105$

The four cases are as follow:

Case I: k is $+ve$, and $1/2 > k > B_2$.

In this case, equation (*a*) has only one root, a_1, between zero and b_1, which corresponds to a minimum value of I. Consequently the distribution of maxima and minima are as shown in Table II (*a*).

Case II: k is $+ve$, and $k < B_2$.

Equation (*a*) will have 3 roots at $\eta = a_1$, a_2 and a_3, where $a_1 < b_1$, and a_2 and a_3 are on either side of b_2. The arrangement of the maxima and minima is tabulated in Table II (*b*).

Case III: k is $-ve$, and $|k| < |B_1|$

In this case, equation (*a*) has two roots, a_1 and a_2, on either side of b_1, which give rise to the arrangement listed in Table II (*c*).

TABLE II

(*a*) Case I ..	Max.	0		b_1				b_3	
	Min.		a_1			b_2			b_5
(*b*) Case II ..	Max.	0		b_1		b_2		b_3	
	Min.		a_1		a_2		a_3		b_4
(*c*) Case III ..	Max.	0		b_1		b_2			b_4
	Min.		a_1		a_2			b_3	
(*d*) Case IV ..	Max.	0				b_2			b_4
	Min.			b_1				b_3	

Case IV: k is $-ve$, and $|k| > |B_1|$

Here, (*a*) has no roots at all, and the maxima and minima occur at $\eta = b_1, b_2, \ldots$ as shown in II (*d*).

The dotted lines are intended to show that the maxima within them from a sort of gradual transition from the case before to the next case. This point will become clear if the intersection points of the graph $y = k$ and $y = J_1(\eta)/\eta$ are studied. In case I, there is no intersection point beyond b_1; but as k decreases, case II becomes operative. At first, a_2 and a_3 are close together, so that the maximum at b_2 will not be prominent. The reverse happens in case III, the first maximum becoming less and less pronounced, until it becomes a minimum in case IV.

It is thus clear that the nature of the ring system depends very much on the value of k, so that even a small fluctuation in the radius of the droplets produces a large change in the disposition of the rings. It is therefore interesting to see how the rings behave as the particle size steadily increases. Suppose that the wavelength λ of the light is a constant. Let us begin with the value $a/\lambda = 2{\cdot}25$ for which $\xi = 3\pi$. This is a case midway between cases II and III. There will be maxima at $\eta = b_1$, b_2, etc., and the rings will appear to be close together. As ξ increases, k becomes negative, case III becomes operative, and the first ring becomes fainter and fainter until it disappears for $|k| > |B_1|$. The whole ring system will now appear to have expanded. This continues for some time, after which k numerically decreases, and the changes take place in the reverse direction, the rings coming close together at $\xi = 4\pi$. Thereafter k is $+ve$, and the second ring becomes more and more indistinct until it disappears when k is $> B_2$. This continues for some time, after which the second ring reappears, the initial stage being reached once more at $\xi = 5\pi$. This finishes one cycle of changes. Another cycle, exactly similar to the above takes place in the range of values 5π to 7π of ξ, except that, in this cycle, case IV cannot be realised at all, since $|\sin(\xi)/\xi|$ is never $> |B_1|$. This continues upto $\xi = 9\pi$, after which case I cannot occur. For droplets of larger radius, *i.e.*, $a/\lambda > 7$, the expanding and contracting of the rings will become less and less noticeable, until, for very large drops, the ring system will correspond to the case of an opaque disk.

In this limiting case of very large drops, $1/\xi^2$ and $\sin(\xi)/\xi$ become intrinsically small, so that (10) reduces to

$$I = K^2 J_1^2(\eta)/\eta^2 \qquad (14)$$

which is the same as the expression for an opaque disk. An interesting point arises in this connection. As shown by equation (34) of the previous

paper by the author, the decrease in the forward intensity due to a large droplet is the same as that for an opaque disk of the same size. Thus, for large droplets, the total energy in the corona must be equal to $2\pi a^2$, or double the amount in the corona due to an opaque disk. However, the expression (14) shows that the corona is identical with that given by an opaque disk of the same size. The discrepancy can be explained by the reason that our theory is not valid for large angles of diffraction. The theory only predicts that for small angles of diffraction, the corona is identical with that produced by an opaque disk. There should be an equal amount of energy diffracted at larger angles, of which no account is taken by the theory.

5. *Numerical Computation*

In order to bring out the facts discussed above in a clear manner, the intensity distribution in the corona has been calculated for a few cases, using

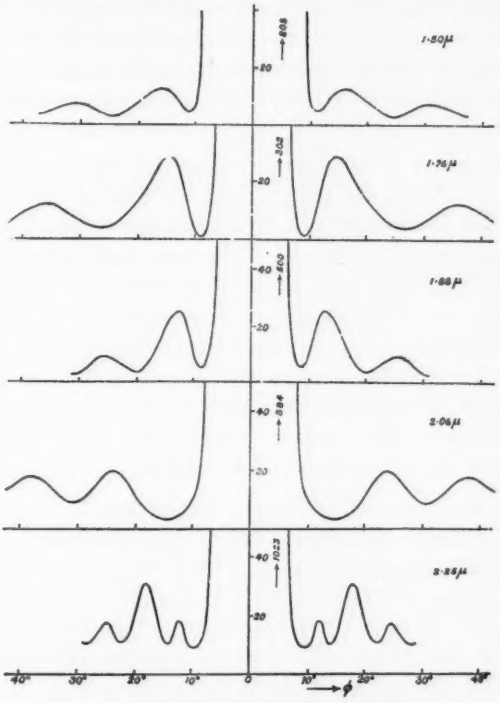

Fig. 2. Intensity Distribution in the Corona

the more accurate expression (7). The wave-length of light was assumed to be 5000 Å.U., and the computation was done for five values of the

radius of the drop. In Fig. 2, the results have been plotted with the angle of diffraction as abscissæ, and the intensity as ordinate. The curves clearly show how the nature of the ring system is altered by changes in the size of drops. They also exhibit the contraction and expansion of the rings as the size of the droplets is increased.

6. Comparison with Experiment

The theory developed in this paper explains most of the observations made by previous workers in the field. Even very early measurements, such as those of Kamtz (Pernter, 1922) showed that the opaque disk theory is only asymptotic in its application, being more and more correct, the larger the size of the drop, as is to be expected from the theory. Mecke (1920 *a*) and Mitra (1928) have observed that the colour sequence in the outer rings, with a white source, had no special order, and that it was very different even with droplets of slight difference in radii. This is easily understood, since our theory shows that, with monochromatic light, the distribution of the rings is highly susceptible to even small changes in a/λ. The case with monochromatic light has also been verified by Mitra, who found marked changes in the position of the rings as the size of the drop changes.

Mecke has given some values of the relative radii of the rings in different cases. If ϕ_1, ϕ_2, ϕ_3 are the angular radii of the first, second and third dark rings in monochromatic light, then $\sin\phi_1 : \sin\phi_2 : \sin\phi_3$ will give the relative values of η for these. They have been tabulated in Table IV of Mecke's paper (1920 *a*). An examination of it will show that the ratio, $\sin\phi_2/\sin\phi_1$, call it ρ_1 varies from 1·8 to nearly 2·4, but that mostly, it is in the neighbourhood of 2·2. The value of $\sin\phi_3/\sin\phi_1$, call it ρ_2, lies between 2·9 and 3·4. Mecke has also found that ρ_1 varies periodically with the radius of the droplet, and that it becomes less than 1·83, the normal value given by the opaque disk theory, only for a small range in each cycle. All these are easily explained by our theory.

Taking one cycle, we start with $\xi = 3\pi$, for which $k = 0$, and $\rho_1 = 1{\cdot}83$, and $\rho_2 = 2{\cdot}8$, which are the normal values. As ξ increases, ρ_1 will slightly diminish and ρ_2 will increase, as may be seen from Fig. 1. This will occur only for a short while, for the maximum at b_1 will soon be too faint, and the ratios will become 2·3 and 3·6 respectively. This holds for case IV also. If we take case II, the ratios will be in the neighbourhood of 2·2 and 3·0, and on passing to case I, they become larger than this. This explains the range of values of ρ_1 and ρ_2 as given by Mecke's experiments as also the fact that ρ_1 is seldom less than 1·83, since it occurs only for a short range after the transition from case II to case III.

It is with great pleasure that I take this opportunity of expressing my gratitude to Prof. Sir C. V. Raman for suggesting the problem, and for the keen and continued interest that he took in the investigation.

Summary

The commonly accepted theory of coronæ, based on the idea that water droplets act as opaque disks, is not only theoretically unsound, but is not also in accord with experimental facts. The theory due to Mecke is also not satisfactory as it based partly on geometrical optics and partly on the theory of diffraction. In this paper, a new theory of the phenomenon is developed, using only the principles of wave-optics, and taking into consideration the portions of the wave-front transmitted through the droplets. The integrals so obtained are integrated by a suitable method, and expressions are obtained for the intensity distribution in the corona. A discussion of these expressions shows that the theory satisfactorily explains most of the phenomena exhibited by coronæ, such as the extreme susceptibility of the ring system to even small changes in the radius of the drops, and the oscillation of the ring system as the radius steadily increases.

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APPENDIX I

Evaluation of the Integral in the expression for X_1

The integral to be evaluated is

$$A = \int_0^{\pi/2} J_0(\eta \sin\theta) \sin(\chi - \xi \cos\theta) \sin\theta \cos\theta \, d\theta$$

The integration is to be done in such a way that the resulting function is readily adapted for numerical computation when ξ is large, *i.e.*, when terms in $1/\xi^2$, $1/\xi^3$, etc., can be neglected compared to those in $1/\xi$.

For the purpose of integration, it is convenient to split the integral into two, and write

$$A = \sin\chi \int_0^{\pi/2} J_0(\eta \sin\theta) \cos(\xi \cos\theta) \sin\theta \cos\theta \, d\theta$$

$$- \cos\chi \int_0^{\pi/2} J_0(\eta \sin\theta) \sin(\xi \cos\theta) \sin\theta \cos\theta d\theta$$

$$= C \sin\chi - S \cos\chi \qquad \text{(say)}$$

Both C and S can be expanded in the form of a series.

Put $\xi \cos\theta = x$, and $\eta \sin\theta = y$.

Then, $\dfrac{dy}{dx} = -\dfrac{\eta \cos\theta}{\xi \sin\theta} = -\dfrac{x}{y}\left(\dfrac{\eta}{\xi}\right)^2$,

Also, the limits of integration are given by

$\theta = 0$, *i.e.*, $x = \xi$, $y = 0$

$\theta = \pi/2$, *i.e.*, $x = 0$, $y = \eta$.

Hence, $$C = \frac{1}{\xi^2} \int_0^{\xi} J_0(y) \cos x \, x \, dx$$

Partially integrating with respect to x,

$$C = \frac{1}{\xi^2}\Big[(\cos x + x \sin x) J_0(y)\Big]_{x=0}^{x=\xi} + \frac{1}{\xi^2} \int_0^{\xi} (\cos x + x \sin x) J_1(y) \, dy$$

Substituting for dy in terms of dx,

$$C = \frac{1}{\xi^2}\left[(\cos\xi + \xi\sin\xi) - J_0(\eta)\right] - \frac{\eta^2}{\xi^4}\int_0^{\xi} (\cos x + x\sin x)\cdot x\cdot J_1(y)/y.dx$$

Again, partially integrating with respect to x, the second term becomes

$$= -\frac{\eta^2}{\xi^4}\left[(3\cos x + 3x\sin x - x^2\cos x)\cdot J_1(y)/y\right]_0^{\xi}$$

$$-\frac{\eta^2}{\xi^4}\int_0^{\xi}(3\cos x + 3x\sin x - x^2\cos x)\cdot\frac{J_2(y)}{y}\,dy$$

Since $\underset{y\to 0}{Lt}\,\dfrac{J_1(y)}{y} = \dfrac{1}{2}$, this is

$$= -\frac{\eta^2}{\xi^4}\left[\tfrac{1}{2}(3\cos\xi + 3\xi\sin\xi - \xi^2\cos\xi) - \frac{3J_1(y)}{y}\right]$$

$$+\frac{\eta^4}{\xi^6}\int_0^{\xi}(3\cos x + 3x\sin x - x^2\cos x)\cdot x\cdot J_2(y)/y^2\cdot dx.$$

Continuing this process, we can finally put C in the form

$$\begin{aligned}
C = {} & \frac{1}{\xi^2}(\cos\xi + \xi\sin\xi) \\
& - \frac{\eta^2}{2\xi^4}(3\cos\xi + 3\xi\sin\xi - \xi^2\cos\xi) \\
& + \frac{\eta^4}{2^2\,2!\,\xi^6}(15\cos\xi + 15\xi\sin\xi - 6\xi^2\cos\xi - \xi^3\sin\xi) \\
& - \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + (-1)^p \frac{\eta^{2p-2}}{2^{p-1}(p-1)!\,\xi^{2p}}(a_{p,p}\cos\xi + a_{p,p-1}\xi\sin\xi \\
& \qquad - a_{p,p-2}\xi^2\cos\xi - \cdots) \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& - \frac{1}{\xi^2}\left[J_0(\eta) - \frac{3\eta^2}{\xi^2}\frac{J_1(\eta)}{\eta} + \frac{15\eta^4}{\xi^4}\frac{J_2(\eta)}{\eta^2} - \cdots\right. \\
& \qquad \left. + (-1)^{p-1} a_{p,p}\left(\frac{\eta}{\xi}\right)^{2p-2}\frac{J_{p-1}(\eta)}{\eta^{p-1}} + \cdots\cdot\right]
\end{aligned}$$

where $a_{p,q} = \dfrac{1}{2^q(q)!}\cdot\dfrac{(p+q)!}{(p-q)!}$ (*Vide* Appendix II)

In an exactly similar manner, S can also be put in the form of a series as below:

$$S = \sum_{p=1}^{\infty} \frac{(-1)^{p-1}\eta^{2p-2}}{2^{p-1}(p-1)!\,\xi^{2p}}\,(a_{p,p}\sin\xi - a_{p,p-1}\,\xi\cos\xi - a_{p,p-2}\,\xi^2\sin\xi + a_{p,p-3}\,\xi^3\cos\xi + \ldots).$$

The above two series give the complete expansion of the integrals C and S. Ordinarily, for the range of particle sizes in which we are interested, ξ is large, and only terms in $1/\xi$ need be taken. Then,

$$A = \sin\chi \cdot \sin\xi/\xi + \cos\chi \cdot \cos\xi/\xi, \text{ so that}$$

$$I = K^2[1/\xi^2 + J_1^2(\eta)/\eta^2 - 2J_1(\eta)/\eta \cdot \sin\xi/\xi].$$

This first approximation is not sufficiently accurate if η is comparable with ξ. Then, the last terms inside the brackets in the right hand side of C must be taken into account. C then becomes

$$C = \frac{\sin\xi}{\xi}\left[1 - \frac{1}{2^2(2)!}\cdot\frac{\eta^4}{\xi^2} + \frac{1}{2^4(4)!}\cdot\frac{\eta^8}{\xi^4} - \cdots\right] + \frac{\cos\xi}{\xi}\left[\frac{\eta^2}{2\xi} - \frac{1}{2^3(3)!}\cdot\frac{\eta^6}{\xi^3} + \cdots\right] = \frac{1}{\xi}\cdot\sin\left(\xi + \frac{\eta^2}{2\xi}\right).$$

Similarly, to a second approximation, $S = \frac{1}{\xi}\cos\left(\xi + \frac{\eta^2}{2\xi}\right)$, so that

$$I = K^2\left[\frac{1}{\xi^2} + \frac{J_1^2(\eta)}{\eta^2} - 2\,\frac{J_1(\eta)}{\eta}\cdot\frac{\sin(\xi + \eta^2/2\xi)}{\xi}\right].$$

If a further approximation were required, it can be obtained by taking all the penultimate terms inside the brackets in the right hand side of C and S. They can also be summed, and lead to the result that, to a third approximation,

$$C = \frac{1}{\xi}\cdot\sin(\xi + \eta^2/2\xi) + \frac{1}{\xi^2}\left[\cos(\xi + \eta^2/2\xi) + \frac{\eta^2}{\xi}\sin(\xi + \eta^2/2\xi) - \frac{\eta^6}{8\xi^3}\cos(\xi + \eta^2/2\xi)\right] \text{ and}$$

$$S = \frac{1}{\xi}\cos(\xi + \eta^2/2\xi) + \frac{1}{\xi^2}\left[\sin(\xi + \eta^2/2\xi) - \frac{\eta^2}{\xi}\cos(\xi + \eta^2/2\xi) - \frac{\eta^6}{8\xi^3}\sin(\xi + \eta^2/2\xi)\right].$$

Any desired degree of approximation can thus be obtained by taking further terms in the expansion, but one more than the second degree will not be needed practically.

APPENDIX II

To prove that $a_{p,q} = (p+q)\,!\,/2^q\,(q)\,!\,(p-q)\,!$

The quantities $a_{p,q}$ depend on the coefficients of the terms in the expansion of the definite integrals, $\int_0^\xi \chi^p \cos x\,dx$, and $\int_0^\xi \chi^p \sin x\,dx$. If we call these respectively C_p and S_p, then

$$C_p = p_0\,\xi^p \sin\xi + p_1\,\xi^{p-1}\cos\xi - p_2\,\xi^{p-2}\sin\xi - \ldots.$$
$$S_p = -p_0\,\xi^p\cos\xi + p_1\,\xi^{p-1}\sin\xi + p_2\,\xi^{p-2}\cos\xi - \ldots.$$

where two terms are $+ve$, the next two are $-ve$ and so on. The series on the right hand side has $(p+1)$ terms each, the last coefficient being p_p. The values of the coefficients are easily seen to be

$$p_q = (p)\,!\,/(p-q)\,!$$

It is to be noted that $p_p = p_{p-1}$.

From the method of partial integration used in Appendix I to determine the series for C, it can be verified that the laws of formation of the numbers $a_{p,q}$ are given by the following:

$$a_{p,0} = p_0\,a_{p-1,0}$$
$$a_{p,1} = (p-1)_0\,a_{p-1,1} + p_1\,a_{p-1,0}$$
$$a_{p,2} = (p-2)_0\,a_{p-1,2} + (p-1)_1\,a_{p-1,1} + p_2\,a_{p-1,0}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{p,q} = (p-q)_0\,a_{p-1,q} + (p-q+1)_1\,a_{p-1,q-1} + \ldots + p_q\,a_{p-1,0}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{p,p-1} = 1_0\,a_{p-1,q-1} + 2_1\,a_{p-1,p-2} + \ldots + p_{p-1}\,a_{p-1,0}$$

and $a_{p,p} = a_{p,p-1}$.

From these substituting the values of p_q, we get,

(1) $a_{p,0} = a_{p-1,0}$, so that $a_{p,0} = 1$, since $a_{1,0} = 1$.

(2) $a_{p,1} = a_{p-1,1} + p\,a_{p-1,0} = a_{p-1,1} + p$. Also, $a_{1,1} = 1$, so that

$$a_{p,1} = \sum_1^p n.$$

(3) $a_{p,2} = a_{p-1,2} + (p-1)\,a_{p-1,1} + p\,(p-1)\,a_{p-1,0}$

$$= a_{p-1,2} + (p-1)\sum_1^{p-1} n + p\,(p-1)$$

$$= a_{p-1,2} + (p-1)\sum_1^p n$$

Hence, $a_{p,2} = a_{2,2} + \sum_3^p (m-1)\sum_1^m n.$

But, $a_{2,2} = a_{2,1} = \sum_{1}^{2} n = (2-1)\sum_{1}^{2} n$, so that

$$a_{p,2} = \sum_{1}^{p}(m-1)\sum_{1}^{m} n.$$

$$(4)\quad a_{p,3} = a_{p-1,3} + (p-2)\,a_{p-1,2} + (p-1)(p-2)\,a_{p-1,1} + p(p-1)(p-2)\,a_{p-1,0}$$

$$= a_{p-1,3} + (p-2)\sum_{1}^{p-1}(m-1)\sum_{1}^{m} n + (p-1)(p-2)\sum_{1}^{p-1} n + p(p-1)(p-2)$$

$$= a_{p-1,3} + (p-2)\sum_{1}^{p}(m-1)\sum_{1}^{m} n.$$

Hence, $a_{p,3} = a_{3,3} + \sum_{4}^{p}(r-2)\sum_{1}^{r}(m-1)\sum_{1}^{m} n.$

But, $a_{3,3} = a_{3,2} = \sum_{1}^{3}(m-1)\sum_{1}^{m} n = (3-2)\sum_{1}^{3}(m-1)\sum_{1}^{m} n$, so that

$$a_{p,3} = \sum_{1}^{p}(r-2)\sum_{1}^{r}(m-1)\sum_{1}^{m} n.$$

Proceeding in this way, it is easily seen that

$$a_{p,q} = \sum_{1}^{p}(t-q+1)\sum_{1}^{t}(s-q+2)\cdots(r-2)\sum_{1}^{r}(m-1)\sum_{1}^{m} n.$$

The expression on the right hand side can be summed up by using the well-known summation:

$$\sum_{x=1}^{n}(a+x)(a+x+1)\cdots(a+x+r-1) = \frac{1}{(r+1)}(a+n)(a+n+1)\cdots(a+n+r).$$

We then get, $\sum_{1}^{m} n = m(m+1)/2 = (m+1)!/2(m-1)!$

$$\sum_{1}^{r}(m-1)\sum_{1}^{m} n = \sum_{1}^{r}(m-1)\,m(m+1)/2 = (r+2)!/2^2(2)!(r-2)!, \text{ etc.}$$

Thus, $a_{p,0} = 1$; $a_{p,1} = (p+1)!/(2)\cdot(p-1)!$;

$a_{p,2} = (p-2)!/2^2(2)!(p-2)!$; $a_{p,3} = (p+3)!/2^3(3)!(p-3)!$;

and so on, so that

$$a_{p,q} = (p+q)!/2^q(q)!(p-q)!$$

It is to be noted that this relation automatically satisfies the equation $a_{p,p} = a_{p,p-1}$.

APPENDIX III

A Note on the Convergency of the Series for C and S

It can be shown that both the integrals C and S, expressed in the form of the series we have chosen, are convergent if $\eta < \xi$. Taking first C, we can express it as follows:

$$\begin{aligned}
C = \sin\xi\,\{&a_{1,0}/\xi - (a_{2,1}\eta^2/2 + a_{3,0}\eta^4/2^2(2)!)/\xi^3 \\
&+ (a_{3,2}\eta^4/2^2(2)!! + a_{4,1}\eta^6/2^3(3)! + a_{5,0}\eta^8/2^4(4)!)/\xi^5 \\
&- \cdots\cdots\cdots\

Substituting for the a's, and also using the relation that, when p is large, $J_{p+1}(\eta)/J_p(\eta) = \eta/2(p+1)$, we get

$$\left|\frac{v_{p+1}}{v_p}\right| = \frac{2p+1}{2p} \cdot \frac{\eta^2}{\xi^2}.$$

Hence, $\underset{p\to\infty}{Lt}\left|\frac{v_{p+1}}{v_p}\right| = \frac{\eta^2}{\xi^2}$, so that J is convergent if $\eta < \xi$.

Thus, the expansion of the integral for X_1 in the form of the series C and S is convergent, and hence valid, if $\eta < \xi$.

BESSEL SUMMATION OF SERIES

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The Bessel method of summation, or the (J_μ, λ) process of summability was first introduced by Dr. S. Minakshisundaram[6] as a means of summing up series of eigenfunctions in boundary-value problems. For the purpose which he then had in view, it was sufficient to prove the fundamental theorems of this process; and the possibility of interrelating it with other known processes remained open. At his instance, I undertook to study that problem, the solution of which is the object of this paper.*

It is proved here that the J_μ process is roughly of the same scope as Cesàro's process. This fact, however, neither vitiates its interest as a generalisation of Riemann's and Lebesgue's methods, nor diminishes its convenience for the treatment of special series.

I owe a great deal to Dr. Minakshisundaram in the way of help, encouragement and criticism.

§ 1. Let $J_\mu(t)$ be the Bessel-function of order μ, so that

$$J_\mu(t) = \frac{t^\mu}{2^\mu \Gamma(\mu+1)} \left\{1 - \frac{t^2}{2(2\mu+2)} + \frac{t^4}{2\cdot 4(2\mu+2)(2\mu+4)} \cdots \right\}.$$

Let

$$a_\mu(t) = 2^\mu \Gamma(\mu+1) \frac{J_\mu(t)}{t^\mu}.$$

Let (λ_n) be a strictly increasing sequence of positive numbers tending to infinity.

Definition.—A series Σa_n is said to be summable (J_μ^K, λ) to the sum S, if the series

$$\sum_{n=0}^{\infty} a_n [a_\mu(\lambda_n t)]^K$$

converges for real $t > 0$, and integral $K > 0$, and if

$$\lim_{t \to 0} \phi_{\mu,K}(t) = s, \text{ exists,}$$

* This paper formed part of a thesis approved for the M.Sc. degree of the University of Madras.

where,

$$\phi_{\mu,K}(t) = \sum_{n=0}^{\infty} a_n \left[a_\mu(\lambda_n t)\right]^K,$$

and

$$\phi_{\mu,1}(t) = \phi_\mu(t).$$

When $K=1$, $\mu=\frac{1}{2}$, $\lambda_n = n$, the definition reads as follows:—

A series Σa_n is said to be summable $(J_{\frac{1}{2}}, n)$ to the sum S, if the series

$$\sum_{n=0}^{\infty} a_n \left(\frac{\sin nt}{nt}\right)$$

converges for real $t > 0$, and if

$$\lim_{t\to 0} \Sigma a_n \left(\frac{\sin nt}{nt}\right) = s,$$

which is Lebesgue's summability.

When $K=2$, $\mu=\frac{1}{2}$, $\lambda_n = n$, the definition reads thus:

A series Σa_n is summable $(J_{\frac{1}{2}}^2, n)$ to the sum S, if the series

$$\sum_{n=0}^{\infty} a_n \left(\frac{\sin nt}{nt}\right)^2$$

converges for real $t > 0$, and if

$$\lim_{t\to 0} \Sigma a_n \left(\frac{\sin nt}{nt}\right)^2 = s,$$

which is Riemann's summability.

When $\mu=\frac{1}{2}$, $\lambda_n = n$, we obtain the generalized Riemann Summability considered by B. Kuttner[5] and M. Vignaux.[8]

The case when $K=1$, $\lambda_n = n$ is important for our purpose, inasmuch as it is closely connected with Cesàro's method of summation, I call this summability J_μ.

§ 2. Minakshisundaram has proved, among others, the following fundamental theorems:

(2·1). *If Σa_n is summable J_μ, then $a_n = o(n^{\mu+\frac{1}{2}})$.*

(2·2). *If Σa_n is summable J_μ, and the convergence of the series $\sum_{n=0}^{\infty} a_n \alpha_\mu(nt)$ is such as to permit integration term by term, then it is summable J_ν, for $\nu > \mu$, to the same sum.*

I prove the following simple theorem:

(2·3). *A necessary and sufficient condition that* Σa_n *which is summable* J_μ *should be summable* $J_{\mu-1}$ *is that the series*

$$t^2 \Sigma n^2 a_n \alpha_{\mu+1}(nt)$$

converges for $t > 0$, *and the sum-function tends to zero with* t. *For,*

$$\phi_\mu(t) - \frac{t^2}{2^2\mu(\mu+1)} \Sigma n^2 a_n \alpha_{\mu+1}(nt)$$

$$= \phi_\mu(t) - 2^{\mu-1}\Gamma(\mu)\, t^2 \Sigma a_n \frac{J_{\mu+1}(nt)}{(nt)^{\mu+1}} n^2.$$

$$= \phi_\mu(t) - 2^{\mu-1}\Gamma(\mu) \Sigma a_n \frac{J_{\mu+1}(nt)}{(nt)^{\mu-1}}$$

$$= 2^{\mu-1}\Gamma(\mu) \Sigma \frac{a_n}{(nt)^{\mu-1}} \left[\frac{2\mu}{nt} J_\mu(nt) - J_{\mu+1}(nt)\right]$$

$$= \Sigma a_n \frac{J_{\mu-1}(nt)}{(nt)^{\mu-1}} \cdot 2^{\mu-1}\Gamma(\mu)$$

$$= \phi_{\mu-1}(t).$$

§ 3. The relation between the J_μ process and the process of Cesàro is established by the following:

THEOREM I. *If a series* Σa_n *is summable* (C, r), *then it is summable* J_μ, *for* $\mu > r + \frac{1}{2}$, *to the same sum.*

Proof. Without loss of generality, the sum of the series may be assumed to be zero.

Let $$S(x) = \sum_{\nu=0}^{n} a_\nu, \text{ for } n \leqslant x < n+1;$$

and $$S^{(r)}(x) = \frac{1}{\Gamma(r)} \int_0^x (x-t)^{r-1} S(t)\, dt;$$

$$S^{(0)}(x) = S(x); \; S = \sum_{0}^{m} a_r = S(m).$$

The Cesàro-summability of order r of the series Σa_n is equivalent to the statement that

$$S^{(r)}(x) = 0(x^r).$$

By Abel's lemma on partial summation,

$$\sum_{0}^{m} a_n \alpha_\mu(nt) = \sum_{0}^{m-1} S_\nu [\alpha_\mu(\nu t) - \alpha_\mu(\overline{\nu+1}\, t)] + S_m \alpha_\mu(mt). \qquad (3\cdot1)$$

Now,

$$\alpha_\mu(\nu t) - \alpha_\mu(\overline{\nu+1}\, t) = -\int_\nu^{\nu+1} \frac{d}{dx} \alpha_\mu(xt) \cdot dx,$$

so that

$$S_\nu [a_\mu (\nu t) - a_\mu (\overline{\nu+1}\, t)] = -\int_\nu^{\nu+1} S^{(0)}(x) \cdot \frac{d}{dx} a_\mu (xt) \cdot dx$$

Hence from (3·1)

$$\sum_0^m a_n a_\mu (nt) = -\sum_{\nu=0}^{m-1} \int_\nu^{\nu+1} S^{(0)}(x) \cdot \frac{d}{dx} a_\mu (xt)\, dx + S_m a_\mu (mt) \qquad (3\cdot2)$$

$$= S_m a_\mu (mt) - \int_0^m S^{(0)}(x) \cdot \frac{d}{dx} a_\mu (xt)\, dx. \qquad (3\cdot3)$$

Integrating the second expression in (3·3) by parts $h+1$ times, where $h = [r]$, we obtain,

$$\sum_0^m a_n a_\mu (nt) = S_m a_\mu (mt) - S^{(1)} \frac{d}{dm} a_\mu (mt) + S^{(2)} \frac{d^2}{dm^2} a_\mu (mt) + \cdots$$
$$+ (-)^{h+1} S^{(h+1)}(m) \frac{d^{h+1}}{dm^{h+1}} a_\mu (mt)$$
$$+ (-)^{h+2} \int_0^m S^{h+1}(x)\, D_x^{h+2} a_\mu (xt)\, dx. \qquad (3\cdot4)$$

By hypothesis, we have,

$$S_m = 0 (m^r)$$
$$S^{(\beta)}(m) = 0 (m^r), \qquad \text{if } \beta \leq r.$$
$$S^{(\beta)}(m) = 0 (m^\beta), \qquad \text{if } \beta > r.$$

Also,

$$* D_x^r [a_\mu (x)] = 0 (x^{-\overline{\mu+\frac{1}{2}}}),$$

for any integer $r > 0$, as $x \to \infty$.

So, if $\gamma > r$,

$$S^{(\gamma)}(m) \frac{d^\gamma}{dm^\gamma} a_\mu (mt) = O (m^{-(\mu+\frac{1}{2})+\gamma})$$
$$= o (1), \text{ if } \mu > \gamma - \tfrac{1}{2}.$$

---

* $J_\mu (x) = O\left(\frac{1}{\sqrt{x}}\right)$ as $x \to \infty$.

So $a_\mu (x) = O\left(\frac{1}{x^{\mu+\frac{1}{2}}}\right)$; $D_x^r a_\mu (x) = \sum_{s=0}^{r} P(s)\, D_x^s J_\mu \cdot D_x^{r-s}\left[\frac{1}{x^\mu}\right]$. But $D_x^s J_\mu$ can be expressed as a linear combination of J_ν's with the index ν ranging from $\mu - s$ to $\mu + s$ Therefore,

$$D_x^s J_\mu = O\left(\frac{1}{\sqrt{x}}\right).$$

Hence

$$D_x^s J_\mu \cdot D_x^{r-s}\left[\frac{1}{x^\mu}\right] = O\left(\frac{1}{x^{\mu+\frac{1}{2}+r-s}}\right) = O\left(\frac{1}{x^{\mu+\frac{1}{2}}}\right)$$

If $\gamma \leqslant r$,

$$S^{(\gamma)}(m)\frac{d^{(\gamma)}}{dm^{\gamma}}a_{\mu}(mt) = O(m^{-(\mu+\frac{1}{2})+r})$$
$$= o(1), \text{ if } \mu > r-\tfrac{1}{2}.$$

Hence, from (3·4), we have

$$\sum_{0}^{m} a_n\, a_{\mu}(nt) = 0(1) + (-1)^{h+2}\int_{0}^{m} S^{(h+1)}(x)\, D_x^{(h+2)}\, a_{\mu}(xt)\, dx,$$
$$\text{if } \mu > h+\tfrac{1}{2}. \qquad (3\cdot5)$$

Now, (3)

$$S^{(h+1)}(m) = S^{(r+\overline{h+1-r})}(m) = \frac{1}{\Gamma(1+h-r)}\int_{0}^{m}(m-x)^{h-r}\, S^{(r)}(x)\, dx,$$

so that

$$\int_{0}^{m} S^{(h+1)}(x)\cdot D_x^{(h+2)}\, a_{\mu}(xt)\cdot dx$$
$$= \frac{1}{\Gamma(1+h-r)}\int_{0}^{m} D_x^{h+2}\, a_{\mu}(xt)\, dx \int_{0}^{x}(x-u)^{h-r}\, S^{(r)}(u)\, du.$$
$$= \frac{1}{\Gamma(1+h-r)}\int_{0}^{m} S^{(r)}(u)\, du \int_{u}^{m}(x-u)^{h-r}\, D_x^{h+2}\, a_{\mu}(xt)\, dx. \quad (3\cdot6)$$

From (3·5), (3·6), we therefore obtain,

$$\sum_{0}^{m} a_n\, a_{\mu}(nt)$$
$$= 0(1) + \frac{(-1)^{h+2}}{\Gamma(1+h-r)}\int_{0}^{m} S^{(r)}(u)\, du \int_{u}^{\infty}(x-u)^{h-r}\, D_x^{h+2}\, a_{\mu}(xt)\, dx, \quad (3\cdot7)$$

provided

$$\int_{0}^{m} S^{(r)}(u)\, du \int_{m}^{\infty}(x-u)^{h-r}\cdot D_x^{h+2}\, a_{\mu}(xt)\, dx \to 0 \text{ as } m \to \infty. \quad (3\cdot8)$$

Now, for a fixed t, we have by the Second Mean Value Theorem,

$$\left|\int_{m}^{\infty}(x-u)^{h-r}\, D_x^{h+2}\, a_{\mu}(xt)\, dx\right|$$
$$\leqslant (m-u)^{h-r}\operatorname*{Max}_{m'>m}\left|\int_{m}^{m'} D_x^{h+2}\, a_{\mu}(xt)\, dx\right|$$
$$= (m-u)^{h-r}\, 0(m^{-\mu-\frac{1}{2}}) \qquad (3\cdot9)$$

and

$$m^{-(\mu+\frac{1}{2})}\int_0^m \left|\, S^{(r)}(u)\,\right| (m-u)^{h-r}\,du = O\left(m^{h+1-(\mu+\frac{1}{2})}\right). \qquad (3\cdot10)$$

$$= 0\,(1), \text{ if } \mu > h+\tfrac{1}{2}.$$

From (3·9) and (3·10), we see that (3·8) is satisfied, and so,

$$\sum_0^m a_n\,\alpha_\mu(nt)$$

$$= 0\,(1) + \frac{(-1)^{h+2}}{\Gamma(1+h-r)}\int_0^m S^{(r)}(u)\,du \int_u^\infty (x-u)^{h-r}\cdot D_x^{h+2}\,\alpha_\mu(xt)\,dx,$$

if $\mu > h+\frac{1}{2}$.

Letting $m \to \infty$, we have,

$$\sum_0^\infty a_n\,\alpha_\mu(nt) =$$

$$\frac{(-1)^{h+2}}{\Gamma(1+h-r)}\int_0^\infty S^{(r)}(u)\,du \int_u^\infty (x-u)^{h-r}\cdot D_x^{h+2}\,\alpha_\mu(xt)\,dx, \qquad (3\cdot11)$$

provided the right side exists.

In order to prove the theorem, we have only to prove that, if

$$\Phi(t) = \int_0^\infty S^{(r)}(u)\,du \int_u^\infty (x-u)^{h-r}\cdot D_x^{h+2}\,\alpha_\mu(xt)\cdot dx, \qquad (3\cdot12)$$

then

(i) $\Phi(t)$ is finite for fixed $t > 0$, (3·13)

and

(ii) $\Phi(t) \to 0$ as $t \to 0$. (3·14)

Setting $x = uy$ in (3·12), we have

$$\Phi(t) = t^{h+2}\int_0^\infty S^{(r)}(u)\,du\cdot u^{1+h-r}\int_1^\infty (y-1)^{h-r}\cdot \alpha_\mu^{(h+2)}(yut)\,dy, \qquad (3\cdot15)$$

where,

$$\alpha_\mu^{h+2}(yut) = \frac{d}{d(yut)}\left[\alpha_\mu^{(h+1)}(yut)\right]$$

Now, let

$$\mathrm{I}(z) = \int_1^\infty (y-1)^{h-r}\, \alpha_\mu^{(h+2)}(yz)\, dy \tag{3·16}$$

so that, from (3·15), we have

$$\Phi(t) = t^{h+2} \int_0^\infty \mathrm{S}^{(r)}(u)\, du \cdot u^{1+h-r}\, \mathrm{I}(ut)\, du \tag{3·17}$$

We shall evaluate the order of $\mathrm{I}(z)$

$$\mathrm{I}(z) = \int_1^{1+1/z} + \int_{1+1/z}^\infty = \mathrm{I}_1 + \mathrm{I}_2, \text{ say.} \tag{3·18}$$

For $|z| \geqslant 1$,

$$\begin{aligned} |\mathrm{I}_1| &\leq \int_1^{1+1/z} |(y-1)^{h-r}\, \alpha_\mu^{h+2}(yz)|\, dy \\ &\leq \mathrm{O}(z^{-\mu-\frac{1}{2}}) \int_1^{1+1/z} (y-1)^{h-r}\, dy \\ &= \mathrm{O}(z^{-(\mu+1\frac{1}{2})-h+r}). \end{aligned} \tag{3·19}$$

And,

$$\begin{aligned} |\mathrm{I}_2| &\leq \int_{1+1/z}^\infty |(y-1)^{h-r}\, \alpha_\mu^{(h+2)}(yz)|\, dy \\ &= \left[\frac{1}{z}\, \alpha_\mu^{(h+1)}(yz)\,(y-1)^{h-r}\right]_{1+1/z}^\infty \\ &\qquad - \frac{h-r}{z} \int_{1+1/z}^\infty \alpha_\mu^{(h+1)}(yz)\,(y-1)^{h-r-1}\, dy \\ &= \mathrm{O}(z^{-(\mu+\frac{1}{2})-h+r}) \end{aligned} \tag{3·20}$$

For $|z| < 1$,

$$\begin{aligned} |\mathrm{I}_1| &= \int_1^{1+1/z} (y-1)^{h-r}\, \alpha_\mu^{(h+2)}(yz)\, dy \\ &= \mathrm{O}(z^{-(h-r+1)}) \end{aligned} \tag{3·21}$$

$$|\mathrm{I}_2| = \mathrm{O}(z^{-(h-r+1)}). \tag{3·22}$$

Hence, from (3·18), (3·19), (3·20) we have, for $|z| \geq 1$,

$$|\mathrm{I}(z)| = \mathrm{O}(z^{-(\mu+1\frac{1}{2})-h+r}) \tag{3·23}$$

and, for $|z| < 1$, from (3·18), (3·21), (3·22),

$$|\mathrm{I}(z)| = \mathrm{O}(z^{-(h-r+1)}). \tag{3·24}$$

Now, from (3·17), we have

$$\Phi(t) = t^{k+2}\left[\int_0^{1/t} S^{(r)}(u)\, u^{1+k-r}\, I(ut)\, du + \int_{1/t}^{\infty}\right]$$

$$= K_1 + K_2, \text{ say.} \tag{3·25}$$

$$K_1 = t^{k+2}\int_0^{1/t} 0(1)\, u^{1+k}\, O\left(\frac{1}{(ut)^{k-r+1}}\right) du, \text{ since } 0 \leqslant u < \frac{1}{t} \tag{3·26}$$

$$K_2 = t^{k+2}\int_0^{\infty} 0(1)\, u^{1+k}\, O\left(\frac{1}{(ut)^{\mu+1+k-r}}\right) du, \text{ since } ut \geqslant 1. \tag{3·27}$$

Now, for a fixed t, K_1 is finite, and K_2 exists if $\mu > r + \frac{1}{2}$, so that $\Phi(t)$ exists for a fixed t.

It is easily seen on the other hand that, if we let $t \to 0$, $K_1 = 0(1)$, $K_2 = 0(1)$, if only $\mu > r + \frac{1}{2}$, so that $\phi(t) \to 0$ as $t \to 0$.

Hence, from (3·11), we observe that

$$\phi_\mu(t) = \sum_{n=0}^{\infty} a_n\, \alpha_\mu(nt)$$

converges for $t > 0$, and $\phi_\mu(t) \to 0$ as $t \to 0$.

Therefore Σa_n is summable J_μ for $\mu > r + \frac{1}{2}$.

Remarks. As a generalization of the above theorem we can prove that *if a series* Σa_n *is summable* (C, α), *then it is summable* (J_μ^k, n) *to the same sum, provided* $K(\mu + \frac{1}{2}) > \alpha + 1$, *where* K *is an integer greater than zero.* The proof follows on exactly similar lines. This theorem includes, as a special case, the following theorem of Kogbetliantz[4]:

If Σa_n *is summable* (C, α) *to the sum* S, *then for* $p > \alpha + 1$, *we have,*

$$\operatorname*{Lim}_{t \to 0} \sum_{n=0}^{\infty} a_n \left(\frac{\sin nt}{nt}\right)^p = s,$$

for, we have only to set $K = p$, $\mu = \frac{1}{2}$ in the above theorems.

Theorem I is 'the best possible result' as can be seen from an example in Fourier series given by Paley.[7]

§ 4. THEOREM II. *If* Σa_n *is summable* J_μ, *and* (i) $\Sigma a_n\, \alpha_\mu(nt)$ *converges in such a way as to permit integration term-by-term, for every* $t > 0$,

(ii) $\Sigma a_n\, \alpha_\mu(nt) = \phi_\mu(t) = O(1)$, *as* $t \to \infty$, *then,* Σa_n *is summable* (C, r) *for* $r > \mu + \frac{1}{2}$.

Proof. Without loss of generality the J_μ sum may be taken to be zero.

$$\text{Let } F(\omega) = \omega^{2\mu+2} \int_0^\infty a_\nu(\omega t)\, \phi_\mu(t)\, t^{2\mu+1}\, dt. \tag{8·1}$$

Then we shall first prove that

$$\lim_{\omega\to\infty} F(\omega) = \lim_{t\to 0} \phi_\mu(t) = 0, \text{ provided } \nu > 2\mu + \frac{3}{2}.$$

Given $\epsilon > 0$, $|\phi_\mu(t)| < \frac{\epsilon}{K}$, for $|t| < \delta(\epsilon)$, where K is a constant to be defined presently.

Now

$$F(\omega) = \omega^{2\mu+2}\left[\int_0^{\delta(\epsilon)} + \int_{\delta(\epsilon)}^\infty\right]$$

$$= I_1 + I_2, \text{ say.}$$

$$|I_1| \leqslant \frac{\epsilon}{K}\cdot \omega^{2\mu+2} \int_0^{\delta(\epsilon)} |a_\nu(\omega t)|\, t^{2\mu+1}\, dt.$$

$$= \frac{\epsilon}{K}\int_0^{\omega\delta(\epsilon)} x^{2\mu+1}\, |a_\nu(x)|\, dx.$$

$$\leqslant \frac{\epsilon}{K}\int_0^\infty x^{2\mu+1}\, |a_\nu(x)|\, dx \tag{8·3}$$

$$< \epsilon, \text{ if } \nu > 2\mu + 1\tfrac{1}{2},$$

where $\int_0^\infty x^{2\mu+1}\, |a_\nu(x)|\, dx = K$, since $a_\nu(x) = O(x^{-(\nu+\frac{1}{2})})$

$$|I_2| < A\int_{\delta\omega}^\infty |a_\nu(x)|\, x^{2\mu+1}\, dx$$

$$< \epsilon,$$

where $|\phi_\mu(t)| < A$, for, given δ we can choose ω so large that

$$\int_{\delta\omega}^\delta < \frac{\epsilon}{A}. \tag{8·4}$$

Hence, from (8·2), (8·3), (8·4), we have

$$F(\omega) = 0(1) \tag{8·5}$$

Next,

$$F(\omega) = \Sigma a_n \, \omega^{2\mu+2} \int_0^\infty t^{2\mu+1} a_\nu(wt) \, a_\mu(nt) \, dt \tag{8·51}$$

$$\omega^{2\mu+2} \int_0^\infty t^{2\mu+1} a_\nu(\omega t) \, a_\mu(nt) \, dt$$

$$= 2^{\mu+\nu} \Gamma(\mu+1) \, \Gamma(\nu+1) \, \omega^{2\mu+2} \int_0^\infty \frac{J_\nu(\omega t) \, J_\mu(nt)}{(\omega t)^\nu \, (nt)^\mu} \, dt$$

$$= K \cdot \frac{\omega^{2\mu+2-\nu}}{n^\mu} \int_0^\infty \frac{J_\nu(\omega t) \, J_\mu(nt)}{t^{\nu-\mu-1}} \, dt, \tag{8·6}$$

where K is some constant.

From the theory of Bessel functions,[9] we have,

$$\int_0^\infty \frac{J_\mu(at) \, J_\nu(bt)}{t^\lambda} \, dt = \frac{b^\nu \, \Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu - \frac{1}{2}\lambda + \frac{1}{2})}{2^\lambda \, a^{\nu-\lambda+1} \, \Gamma(\nu+1) \, \Gamma(\frac{1}{2}\lambda + \frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2})}$$

$$\times {}_2F_1\left(\frac{\mu+\nu-\lambda+1}{2}, \frac{\nu-\lambda-\mu+1}{2}, \nu+1; \frac{b^2}{a^2}\right)$$

provided, $0 < b < a$, $R(\mu+\nu+1) > R(\lambda) > -1$. (8·7)

From (8·7) we deduce that

$$\int_0^\infty \frac{J_\nu(\omega t) \, J_\mu(nt)}{t^{\nu-\mu-1}} \, dt = \frac{n^\mu}{2^{\nu-\mu-1} \, \omega^{2\mu+2-\nu} \, \Gamma(\nu-\mu)} \left(1 - \frac{n^2}{\omega^2}\right)^{\nu-\mu-1},$$

$$\text{if } 0 < n < \omega,$$

and $$= 0, \text{ if } n > \omega. \tag{8·8}$$

From (8·5), (8·5 1), (8·6) and (8·8), we observe that

$$K \cdot F(\omega) = \sum_{n \leqslant \omega} a_n \left(1 - \frac{n^2}{\omega^2}\right)^{\nu-\mu-1} \to 0 \text{ as } \omega \to \infty \tag{8·9}$$

where K is a constant.

Hence Σa_n is summable $(R; n^2, \nu-\mu-1)$ for $\nu > 2\mu + 1\frac{1}{2}$. By the second theorem of consistency for summable series due to Hardy,[2] Σa_n is summable $(R; n, \nu-\mu+1)$; that is, Σa_n is summable (C, r), for $r > \mu + \frac{1}{2}$.

Remarks. As a generalization of Theorem II, we can prove that:

If (i) the series $\Sigma a_n \, a_\mu(\lambda_n t)$ converges for all $t > 0$, and uniformly in every finite interval, and

(ii) $\phi_\mu(t)$ is bounded as $t \to \infty$, and

(iii) $\phi_\mu(t) \to s \neq \infty$ as $t \to 0$,

then Σa_n is summable (R; λ, K), for $K > \mu + \frac{1}{2}$.

It must be stated, in passing, that Theorem II can be put in the following form in which it finds application usually, *viz.*,

If Σa_n is summable J_μ, then it is summable (*C*, r) *for* $r > \mu + 1\frac{1}{2}$.

This follows easily, since $a_n = 0\,(n^{\mu+\frac{1}{2}})$, and so $\Sigma a_n\, a_\nu\,(nt)$ converges uniformly and absolutely for $\nu > \mu + 1$. But this is not the best possible result. The restriction on r seems to admit further relaxation.

However, Theorem II is 'the best possible result' as can be seen from the classical example of Hahn[1] in Fourier series.

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INDEX TO VOL. XVII (A)

AUTHORS' INDEX

Chandra Sekharan, K. .. Bessel summation of series, 219.

Deliwala, C. V., and Shah, N. M. — Aluminium chloride, a new reagent for the condensation of β-ketonic esters with phenols, Part VI, 7.

Doss, K. S. Gururaja and Rao, Basrur Sanjiva — Pulsations on mercury surface caused by polarisation, 158.

Gogate, V. S. .. *See* Prasad and Gogate.

Kartha, A. R. Sukumaran, and Menon, K. N. — The isolation and constitution of an acid from the root bark of *Ixora coccinea* (Linn.), 11.

.. A tentative method for the estimation of mono-unsaturated and di-unsaturated glycerides, 114.

Levi, F. W. .. Contributions to the theory of ordered groups, 199.

Menon, K. N. .. *See* Kartha and Menon.

Menon, P. Kesava .. Some theorems on residues, 107.

Murthy, G. V. L. N., and Seshadri, T. R. — Raman effect and hydrogen bonds, Part VIII, 55.

Nagabhushanam, K. .. *See* Rao and Nagabhushanam.

Neelakantam, K., Rao, P. Suryaprakasa and Seshadri, T. R. — Chemical components of Indian Tulip (*Thespasia* and *populnea*) Flowers, 26.

Pandya, Kantilal C., and Pandya, (Miss) Rashmi Bala — Condensation of malonanilic acid with aldehydes, Part V, 1.

Pandya, (Miss) Rashmi Bala — *See* Pandya and Pandya.

Pillai, K. C. Sreedharan . Trend Analyser, 187.

Pillai, S. S. .. Lattice points in a right-angled triangle, II, III, 58, 62.

.. On $\sigma_{-1}(n)$ and $\phi(n)$, 67.

Prasad, Mata and Gogate, V. S. — Opacity changes in gel-forming mixtures during setting, Part I, 161.

Ramachandran, G. N. .. On the transmission of light through a cloud of randomly distributed particles, 171.

.. The theory of coronæ and of iridescent clouds, 202.

Rangaswami, S., and Seshadri, T. R. — Constitution of kanugin, Part I, 20.

Rao, Basrur Sanjiva .. *See* Doss and Rao.

Rao, Kittur Subba, and Thiruvenkatachar, V. R. — Hysteresis in sorption, X, 46.

Rao, K. Sambasiva and Nagabhushanam, K. — A note on Vinogradow numbers, 71.

Rao, N. S. Subba .. Atmospherics during the monsoon period, 83.

Rao, P. Suryaprakasa .. *See* Neelakantam and others.

Rao, P. Suryaprakasa, and Seshadri, T. R. — The structural chemistry of naturally occurring flavones and flavonols, 119.

Sen, Sailesh Chandra .. *See* Singh and Sen.

Seshadri, T. R. .. *See* Rangaswami and Seshadri.

.. *See* Neelakantam and others.

.. *See* Murty and Seshadri.

.. *See* Rao and Seshadri.

.. Richard Willstätter, 143.

Seshadri, T. R., and Venkateswarlu, V. — Synthetic experiments in the benzopyrone series, Part VII, 16.

Shah, N. M. .. *See* Deliwala and Shah.

Singh, Bawa Karthar, and Sen, Sailesh Chandra — Studies on the dependence of optical rotatory power on chemical constitution, Part XXII, 33.

Thiruvenkatachar, V. R. .. *See* Rao and Thiruvenkatachar.

Vaidyanathaswamy, R. .. On disjunction in intuitionist logic, 41.

Varier, N. S. .. Fixed oil from the seeds of *Gnetum scandens* (Roxb.), 195.

Venkatarayudu, T. .. Normal co-ordinates of symmetric point groups, I, II, 50, 75.

.. The character table of a sub-group of the symmetric group of degree, 8, 79.

Venkateswarlu, V. .. *See* Seshadri and Venkateswarlu.

INDEX TO VOL. XVII (A)

AUTHORS' INDEX

Chandra Sekharan, K. .. Bessel summation of series, 219.

Deliwala, C. V., and Shah, N. M. — Aluminium chloride, a new reagent for the condensation of β-ketonic esters with phenols, Part VI, 7.

Doss, K. S. Gururaja and Rao, Basrur Sanjiva — Pulsations on mercury surface caused by polarisation, 158.

Gogate, V. S. .. *See* Prasad and Gogate.

Kartha, A. R. Sukumaran, and Menon, K. N. — The isolation and constitution of an acid from the root bark of *Ixora coccinea* (Linn.), 11.

.. A tentative method for the estimation of mono-unsaturated and di-unsaturated glycerides, 114.

Levi, F. W. .. Contributions to the theory of ordered groups, 199.

Menon, K. N. .. *See* Kartha and Menon.

Menon, P. Kesava .. Some theorems on residues, 107.

Murthy, G. V. L. N., and Seshadri, T. R. — Raman effect and hydrogen bonds, Part VIII, 55.

Nagabhushanam, K. .. *See* Rao and Nagabhushanam.

Neelakantam, K., Rao, P. Suryaprakasa and Seshadri, T. R. — Chemical components of Indian Tulip (*Thespasia* and *populnea*) Flowers, 26.

Pandya, Kantilal C., and Pandya, (Miss) Rashmi Bala — Condensation of malonanilic acid with aldehydes, Part V, 1.

Pandya, (Miss) Rashmi Bala — *See* Pandya and Pandya.

Pillai, K. C. Sreedharan . Trend Analyser, 187.

Pillai, S. S. .. Lattice points in a right-angled triangle, II, III, 58, 62.

.. On $\sigma_{-1}(n)$ and $\phi(n)$, 67.

Prasad, Mata and Gogate, V. S. — Opacity changes in gel-forming mixtures during setting, Part I, 161.

Ramachandran, G. N. .. On the transmission of light through a cloud of randomly distributed particles, 171.

.. The theory of coronæ and of iridescent clouds, 202.

Rangaswami, S., and Seshadri, T. R. Constitution of kanugin, Part I, 20.

Rao, Basrur Sanjiva .. *See* Doss and Rao.

Rao, Kittur Subba, and Thiruvenkatachar, V. R. Hysteresis in sorption, X, 46.

Rao, K. Sambasiva and Nagabhushanam, K. A note on Vinogradow numbers, 71.

Rao, N. S. Subba .. Atmospherics during the monsoon period, 83.

Rao, P. Suryaprakasa .. *See* Neelakantam and others.

Rao, P. Suryaprakasa, and Seshadri, T. R. The structural chemistry of naturally occurring flavones and flavonols, 119.

Sen, Sailesh Chandra .. *See* Singh and Sen.

Seshadri, T. R. .. *See* Rangaswami and Seshadri.

.. *See* Neelakantam and others.

.. *See* Murty and Seshadri.

.. *See* Rao and Seshadri.

.. Richard Willstätter, 143.

Seshadri, T. R., and Venkateswarlu, V. Synthetic experiments in the benzopyrone series, Part VII, 16.

Shah, N. M. .. *See* Deliwala and Shah.

Singh, Bawa Karthar, and Sen, Sailesh Chandra Studies on the dependence of optical rotatory power on chemical constitution, Part XXII, 33.

Thiruvenkatachar, V. R. .. *See* Rao and Thiruvenkatachar.

Vaidyanathaswamy, R. .. On disjunction in intuitionist logic, 41.

Varier, N. S. .. Fixed oil from the seeds of *Gnetum scandens* (Roxb.), 195.

Venkatarayudu, T. .. Normal co-ordinates of symmetric point groups, I, II, 50, 75.

.. The character table of a sub-group of the symmetric group of degree, 8, 79.

Venkateswarlu, V. .. *See* Seshadri and Venkateswarlu.

TITLE INDEX

Aluminium chloride, a new reagent for the condensation of β-ketonic esters with phenols, VI (Deliwala and Shah), 7.

Atmospherics during the monsoon period (Rao), 83.

Benzopyrone series, synthetic experiments, VII (Seshadri and Venkateswarlu), 16.

Bessel summation of series (Chandra Sekharan), 219.

Coronæ and iridescent clouds, the theory (Ramachandran), 202.

Disjunction in intuitionist logic (Vaidyanathaswamy), 41.

Flavones and flavonols, naturally occurring, the structural chemistry (Rao and Seshadri), 119.

Glycerides, mono-unsaturated and di-unsaturated, a tentative method for the estimation (Kartha and Menon), 114.

Gnetum scandens (Roxb.), seeds of, fixed oil (Varier), 195.

Groups, ordered, contributions to the theory (Levi), 199.

Hysteresis in sorption, X (Rao and Thiruvenkatachar), 46.

Ixora coccinea (Linn.), root bark of, the isolation and constitution of an acid from (Kartha and Menon), 11.

Kanugin, constitution, I (Rangaswami and Seshadri), 20.

Lattice points in a right-angled triangle, II, III (Pillai), 58, 62.

Light, transmission, through a cloud of randomly distributed particles (Ramachandran), 171.

Malonanilic acid, condensation, with aldehydes, V (Pandya and Pandya), 1.

Normal co-ordinates of symmetric point groups, I, II (Venkatarayudu), 50, 75.

Opacity changes in gel-forming mixtures during setting, I (Prasad and Gogate 161.

Optical rotatory power, dependence, on chemical constitution, studies, XXII (Singh and Sen), 33.

Pulsations on mercury surface caused by polarisation (Doss and Rao), 158.

Raman effect and hydrogen bonds, VIII (Murty and Seshadri), 55.

Residues, some theorems (Menon), 107.

(*Thespasia populnea*), Indian Tulip, flowers, chemical components (Neelakantam and others), 26.

Trend analyser (Pillai), 187.

$\sigma_{-1}(n)$ and $\phi(n)$ (Pillai), 67.

Symmetric group of degree 8, a sub-group of, the character table (Venkatarayudu), 79.

Vinogradow numbers, a note (Rao and Nagabhushanam), 71.

Willstätter, Richard (Seshadri), 143.

TITLE INDEX

Aluminium chloride, a new reagent for the condensation of β-ketonic esters with phenols, VI (Deliwala and Shah), 7.

Atmospherics during the monsoon period (Rao), 83.

Benzopyrone series, synthetic experiments, VII (Seshadri and Venkateswarlu), 16.

Bessel summation of series (Chandra Sekharan), 219.

Coronæ and iridescent clouds, the theory (Ramachandran), 202.

Disjunction in intuitionist logic (Vaidyanathaswamy), 41.

Flavones and flavonols, naturally occurring, the structural chemistry (Rao and Seshadri), 119.

Glycerides, mono-unsaturated and di-unsaturated, a tentative method for the estimation (Kartha and Menon), 114.

Gnetum scandens (Roxb.), seeds of, fixed oil (Varier), 195.

Groups, ordered, contributions to the theory (Levi), 199.

Hysteresis in sorption, X (Rao and Thiruvenkatachar), 46.

Ixora coccinea (Linn.), root bark of, the isolation and constitution of an acid from (Kartha and Menon), 11.

Kanugin, constitution, I (Rangaswami and Seshadri), 20.

Lattice points in a right-angled triangle, II, III (Pillai), 58, 62.

Light, transmission, through a cloud of randomly distributed particles (Ramachandran), 171.

Malonanilic acid, condensation, with aldehydes, V (Pandya and Pandya), 1.

Normal co-ordinates of symmetric point groups, I, II (Venkatarayudu), 50, 75.

Opacity changes in gel-forming mixtures during setting, I (Prasad and Gogate 161.

Optical rotatory power, dependence, on chemical constitution, studies, XXII (Singh and Sen), 33.

Pulsations on mercury surface caused by polarisation (Doss and Rao), 158.

Raman effect and hydrogen bonds, VIII (Murty and Seshadri), 55.

Residues, some theorems (Menon), 107.

(*Thespasia populnea*), Indian Tulip, flowers, chemical components (Neelakantam and others), 26.

Trend analyser (Pillai), 187.

$\sigma_{-1}(n)$ and $\phi(n)$ (Pillai), 67.

Symmetric group of degree 8, a sub-group of, the character table (Venkatarayudu), 79.

Vinogradow numbers, a note (Rao and Nagabhushanam), 71.

Willstätter, Richard (Seshadri), 143.